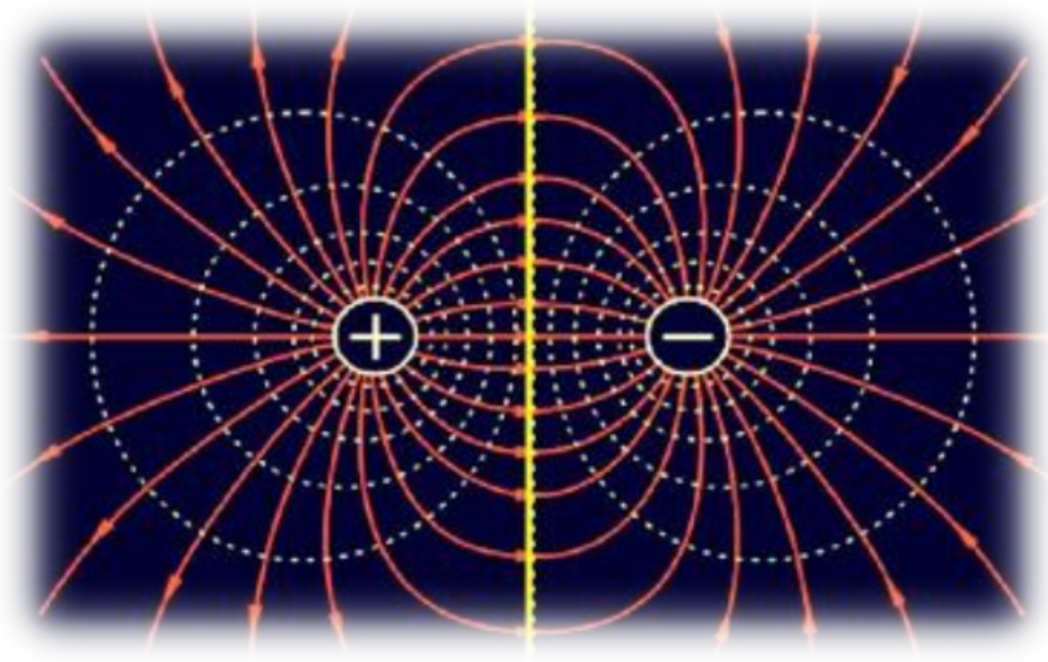
General Physics:
Electricity & Magnetism

Shuai Dong



A legend of inventor, flagman of the 2nd industrial revolution

Thomas Alva Edison，1847-1931

The stories before Edison!

Chapter 1: electrostatic field

1. Static electricity
2. Electric field
3. Gauss's law
4. Electric potential
5. Electrostatic energy

Chapter 1: Homework

1. 9-2, 9-3, 9-5, 9-11, 9-14
2. Others in the ppt
3. Deadline: Friday, April 26

1.1 Static electricity

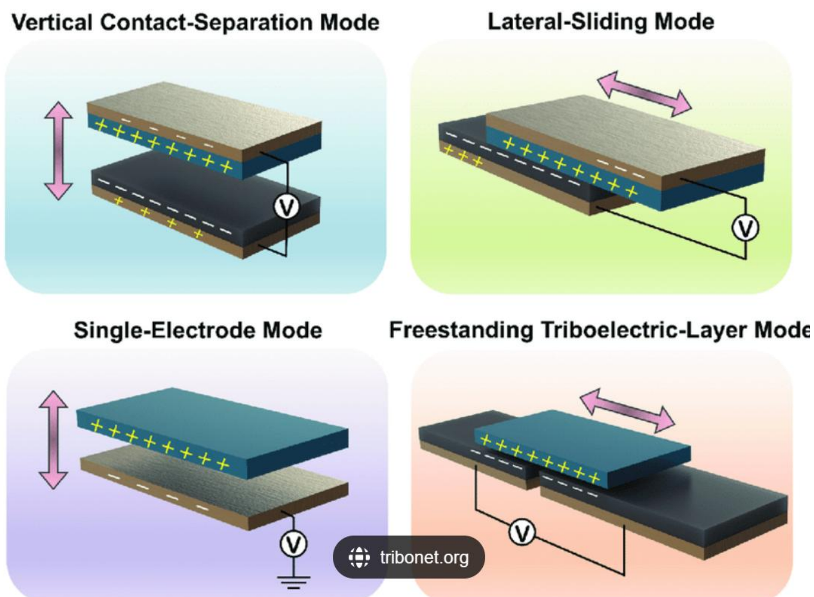
Electrification by friction

Electroscope

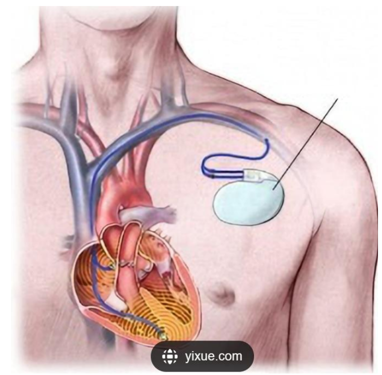
invented in the 18th century

1.1 Static electricity

Triboelectric nanogenerator (TENG)



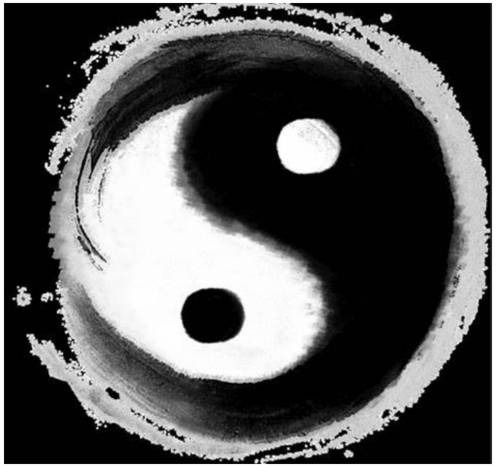
Bioelectronic appliations



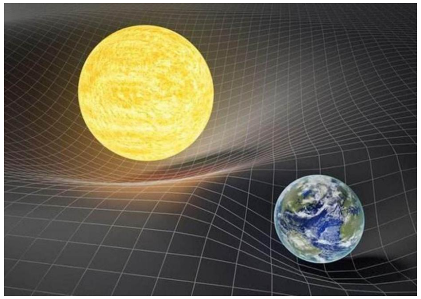
Blue energy

补充阅读：摩擦纳米发电机真的有前景吗？

1.1 Static electricity

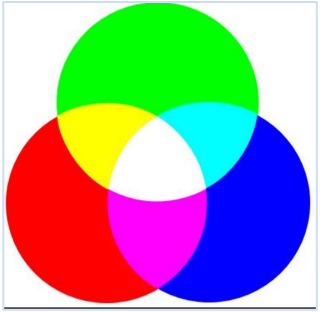


Charge
binary elements:
positive (+) <-----> negative (-)
force: ++/--: repulsive
+-: attractive



Gravitation

Unsigned element
mass



Color

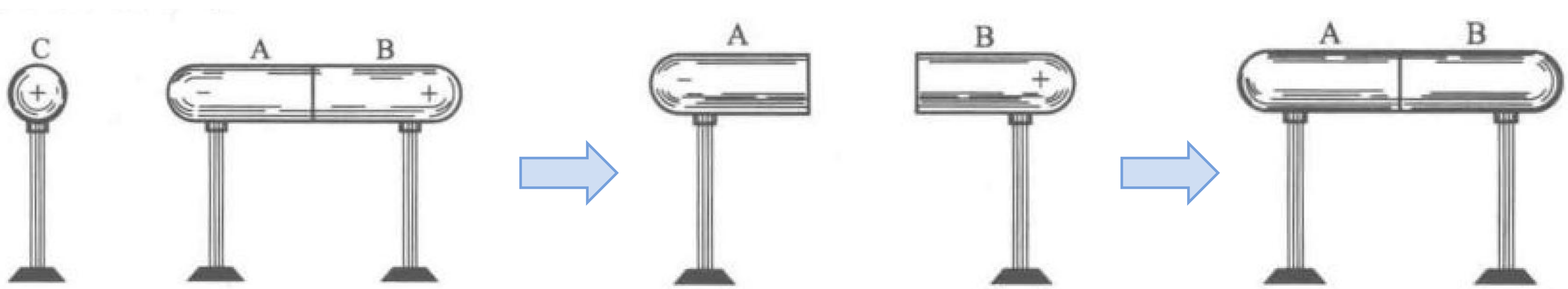
Three primary elements
Red+Green+Blue

1.1 Static electricity

Invariance of Charge:

Net charge Q= positive charge Q_+- negative charge Q_-

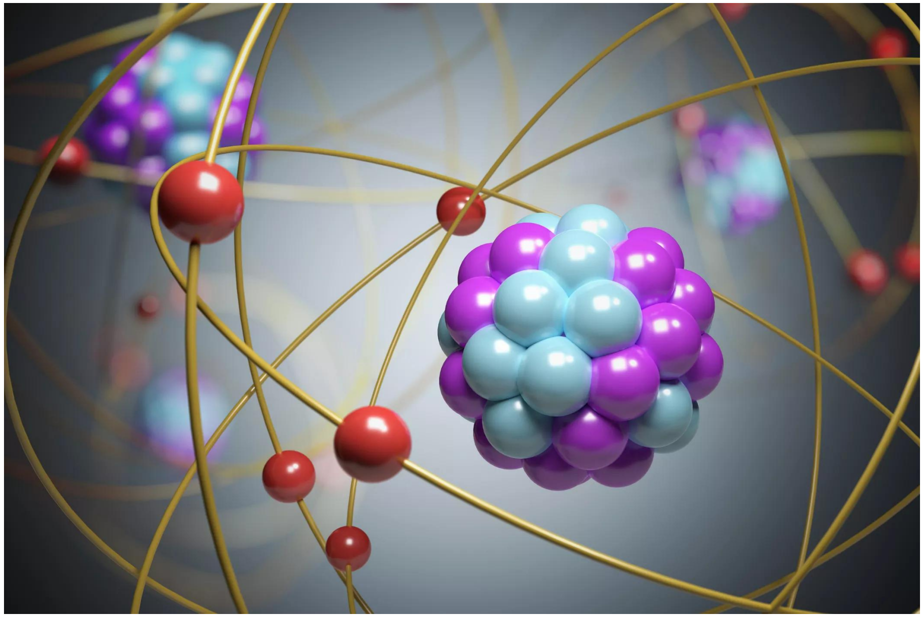
If Q=0 ----> charge neutral or electric neutral



electrostatic induction

Unit of charge: Coulomb (C)
named after Charles-Augustin de Coulomb
(1736-1806)

1.1 Static electricity



Atom = nucleus + electron(s)
nucleus = proton(s) +neutron(s)

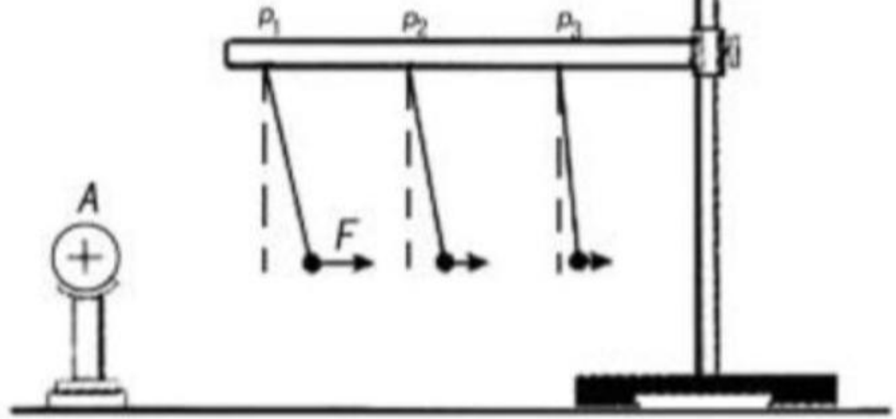
- proton: $+e$
- electron: $-e$
- neutron: 0
- e: elementary charge
- $=1.602 \times 10^{-19}$ C
- $Q = ne$ (n: an integer)

Conductor: many free carriers (+e or -e, from electrons/hole, ions, plasma)
Insulator: no free carriers
Semiconductor: a few carriers

1.1 Static electricity

torsion balance
1785

Coulomb's law



$\mathbf{F}_{21} = kq_1q_2\mathbf{r}_{21}/r_{21}^3$ $\mathbf{r}_{21}$: vector

$k = 1/4\pi\varepsilon_0 = 9\times10^9$ Nm²/C²

$\varepsilon_0 = 8.85\times10^{-12}$ C²/(Nm²)

ε_0: permittivity of vaccum or dielectric constant of vacuum

1.1 Static electricity

Current $I=Q/T$ (SI units: Ampère, Coulomb, second)

1 A= 1 C/1 s SI: Système International d'Unités

Charge of a power bank

$Q = 20000 \times 10^{-3}$ A x 3600 s

$= 7.2 \times 10^4$ C

$n=Q/e=4.5 \times 10^{23} = 0.75\ N_A$

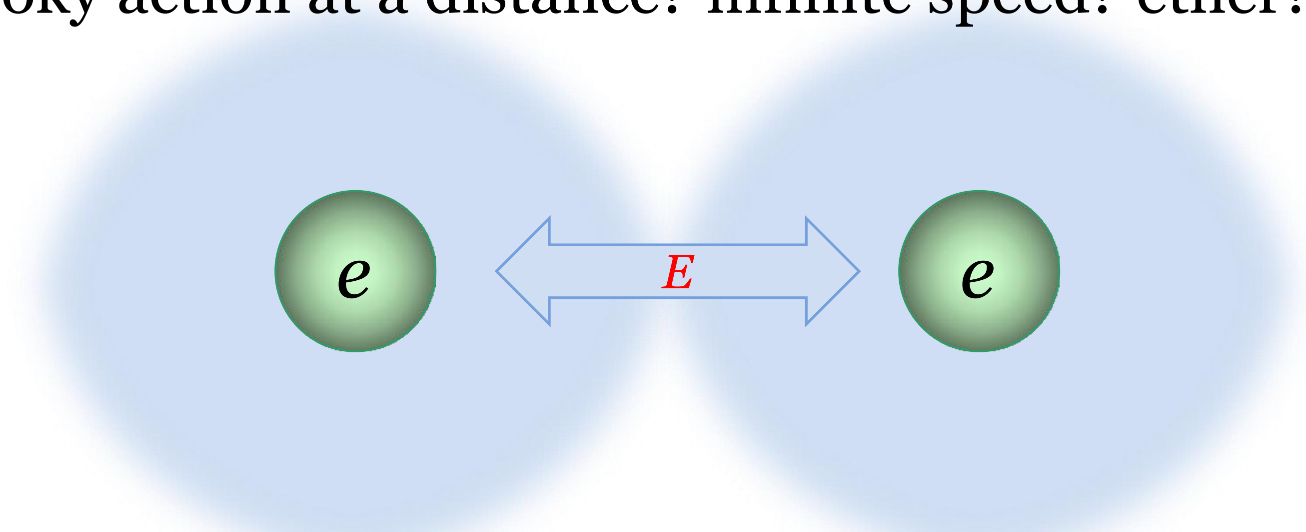
N_A: Avogadro constant

Li^+ ions: 0.75 mol x 7 g/mol = 5.25 g

1.2 Electric field

Contact force *vs* non-contact force

Spooky action at a distance? infinite speed? ether?

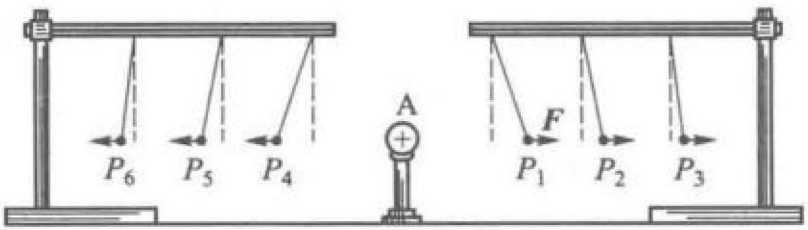


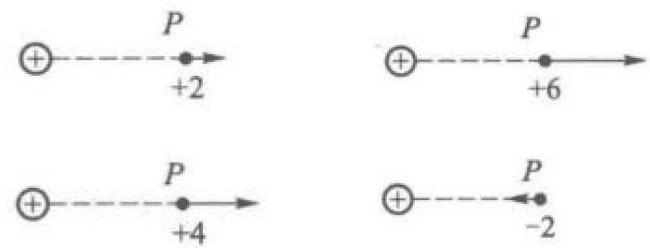
invisible matter

Answer: electric field + finite speed (c)

1.2 Electric field

Electric field **E** is defined as **F**/*q*, which is a vector.





The probe charge:

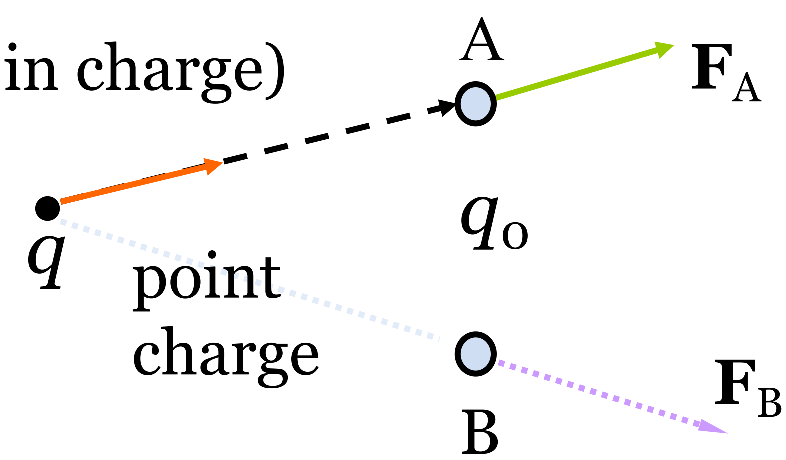
1) point charge (small enough in dimensions);

2) weak charge (small enough in charge)

Unit: N/C

$\mathbf{F}_{21}=kq_1q_2\mathbf{r}_{21}/r_{21}^3$

$\mathbf{E}_{21}=kq_2\mathbf{r}_{21}/r_{21}^3$



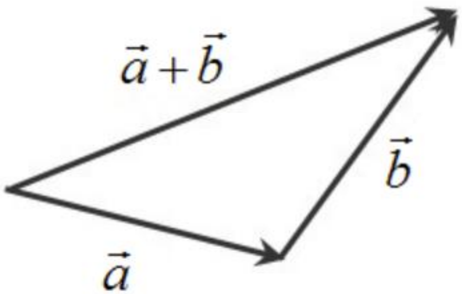
1.2 Electric field

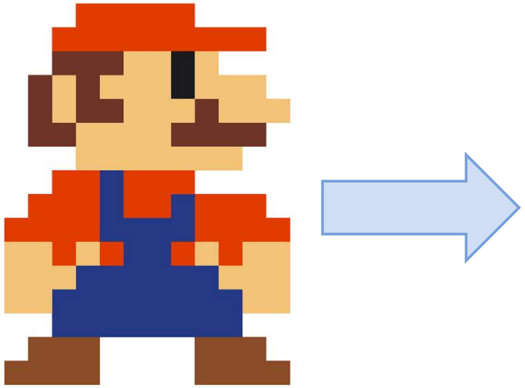
Vector superposition of forces (fields)

$\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2+\ldots+\mathbf{F}_n=\sum_i \mathbf{F}_i$

$\mathbf{E}=\mathbf{F}/q=\sum_i \mathbf{F}_i/q=\sum_i \mathbf{E}_i$

for a set of discrete charged points



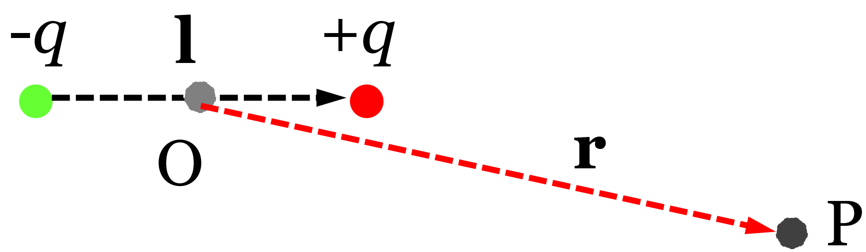


$\mathbf{E}=1/(4\pi\varepsilon_0)\int_q \mathbf{r}\,dq/r^3$

for continuous distribution of charge

1.2 Electric field

Example 1: the field of an electric dipole



Two point charges:

+q & -q with distance **l**

dipole moment **p**=q**l**

Case 1: **r**||**l**

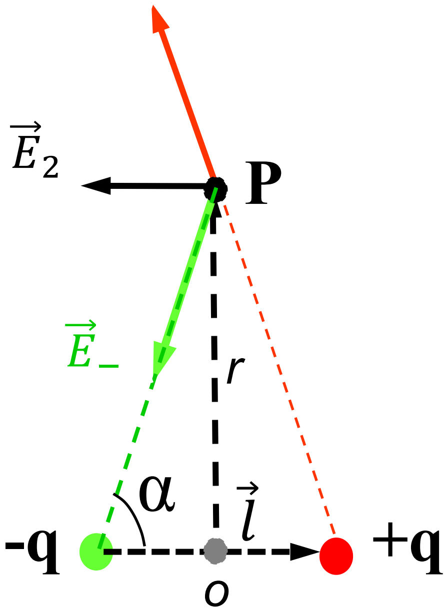
$$E_+ = \frac{q}{4\pi\varepsilon_0 (r-\frac{l}{2})^2} \qquad E_- = \frac{-q}{4\pi\varepsilon_0 (r+\frac{l}{2})^2}$$

$$E_p = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r-\frac{l}{2})^2} - \frac{1}{(r+\frac{l}{2})^2}\right] = \frac{q}{4\pi\varepsilon_0}\frac{2rl}{[r^2-(\frac{l}{2})^2]^2}$$

In the limit of $r>>l$

$$E_p = \frac{1}{4\pi\varepsilon_0}\frac{2ql}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3}$$

1.2 Electric field



Case 2: **r⊥l**

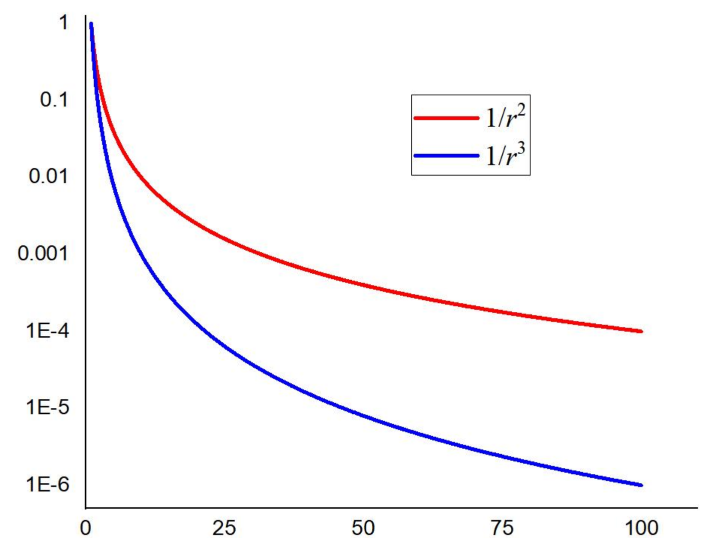
$$E_+ = E_- = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2 + (\frac{l}{2})^2}$$

$$E_2 = -2E_+\cos\alpha \qquad \cos\alpha = \frac{l/2}{\sqrt{r^2 + (l/2)^2}}$$

$$E_2 = \frac{-q}{4\pi\varepsilon_0}\frac{l}{[r^2 + (\frac{l}{2})^2]^{3/2}}$$

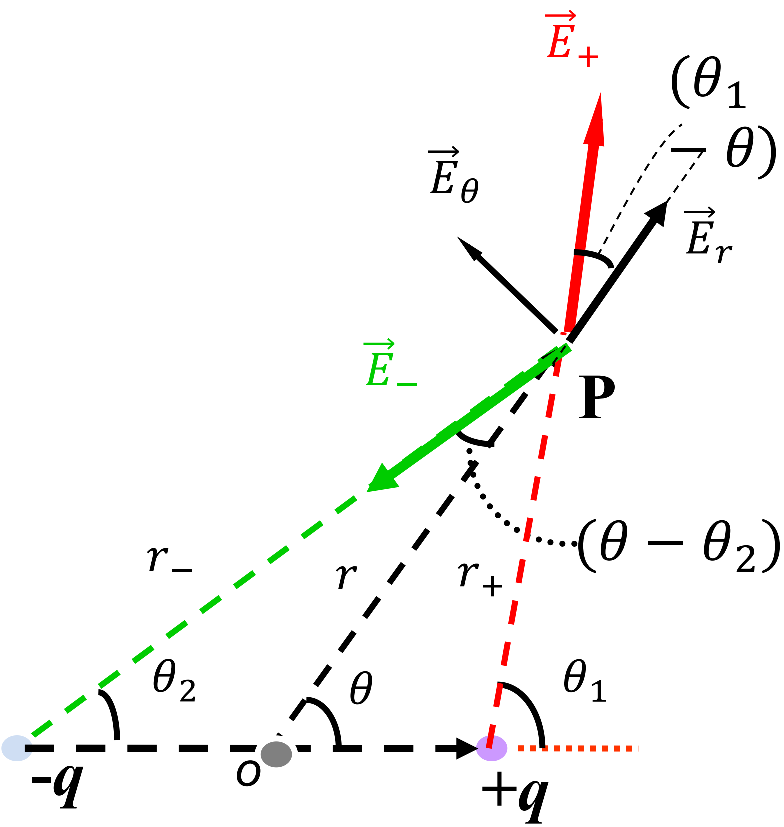
In the limit of *r>>l*

$$E_p = \frac{1}{4\pi\varepsilon_0}\frac{-ql}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{-p}{r^3}$$



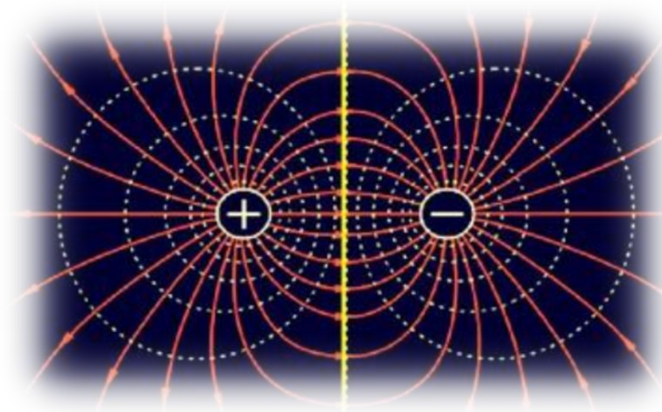
1.2 Electric field

Case 3: other arbitrary **l**



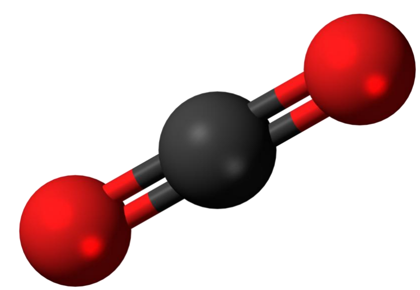
In the limit of **r>>l**

$E_p \propto p/r^3$



Additional Homework 1: comparison of electric field generated by H_2O and CO_2 molecules

H_2O



CO_2

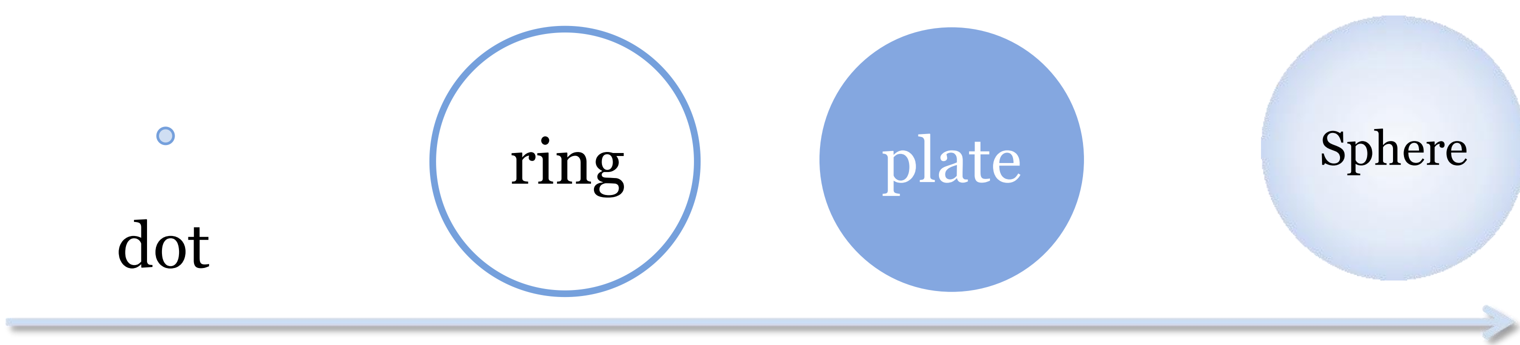
1.2 Electric field

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_q \frac{dq}{r^3}\mathbf{r}$$

1) Linear density η=dq/dl, dl : line element

2) Surface density σ=dq/ds, ds: surface element

3) Volume density ρ=dq/dv, dv: volume element

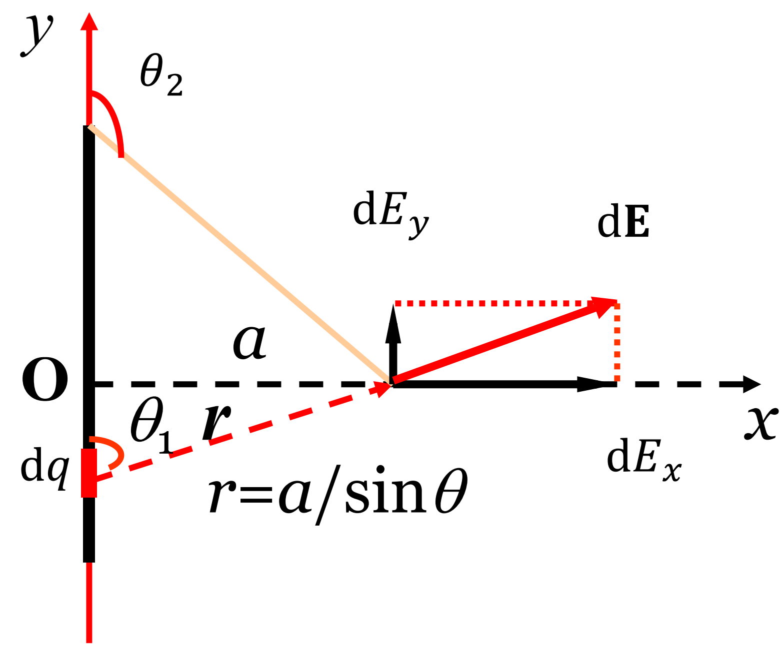


from 0D to 3D

1.2 Electric field

Example 2: a charged rod with linear density η, a point with distance a from the rod: $\mathrm{d}q=\eta \mathrm{d}y$;

$d\mathbf{E}=\mathbf{r}dq/(4\pi\varepsilon_0 r^3)$



$dE_x=\mathbf{E}\sin\theta \qquad dE_y=\mathbf{E}\cos\theta$

$y=-a\ ctg\theta$ ---> $dy=ad\theta/\sin^2\theta$

$E_x=\int\sin\theta dq/(4\pi\varepsilon_0 r^2)=\int\eta\sin\theta dy/(4\pi\varepsilon_0 r^2)=\int\eta\sin\theta d\theta/(4\pi\varepsilon_0 a)$

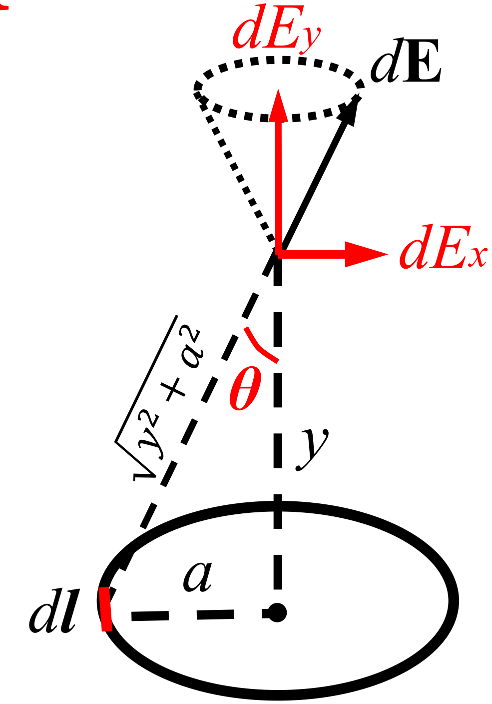
$=\eta(\cos\theta_1-\cos\theta_2)/(4\pi\varepsilon_0 a) \qquad E_y=\eta(\sin\theta_2-\sin\theta_1)/(4\pi\varepsilon_0 a)$

For a infinite long rod, $\theta_1=0^o$, $\theta_2=180^o$

$E_x=\eta/(2\pi\varepsilon_0 a) \qquad E_y=0$

1.2 Electric field

Example 3: For a charged circle with linear density $\eta = q/2\pi a$



For any point at the central axis, $E_x = 0$, which is protected by symmetry

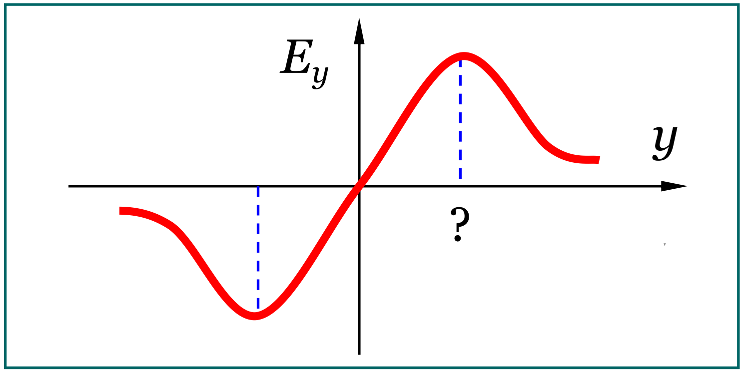
$$E_y = \int dE_y = \int \cos\theta \, dE \qquad dE = \eta dl / [4\pi\varepsilon_0 (y^2 + a^2)]$$

$$\cos\theta = y/(y^2 + a^2)^{1/2}$$

$$E_y = \int y\eta dl / [4\pi\varepsilon_0 (y^2 + a^2)^{3/2}]$$

$$= 2\pi a y \eta / [4\pi\varepsilon_0 (y^2 + a^2)^{3/2}]$$

$$= qy / [4\pi\varepsilon_0 (y^2 + a^2)^{3/2}]$$



For $a \ll y$, $\mathbf{E} = q\boldsymbol{y}/(4\pi\varepsilon_0 y^3)$, back to the point charge limit

1.2 Electric field

$E_y = qy/[4\pi\varepsilon_0(y^2+a^2)^{3/2}]$

To find the maximum E_y:

$dE_y/dy \sim (y^2+a^2)^{-3/2} - 3y^2(y^2+a^2)^{-5/2}$

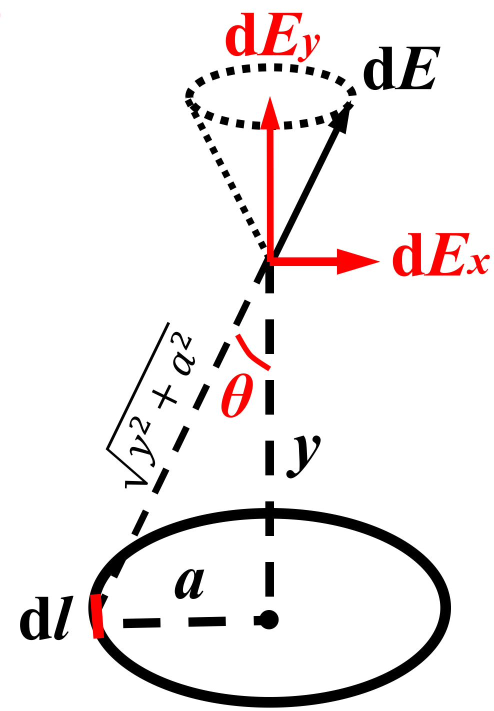
$= (y^2+a^2)^{-3/2}[1-3y^2/(y^2+a^2)] = 0$

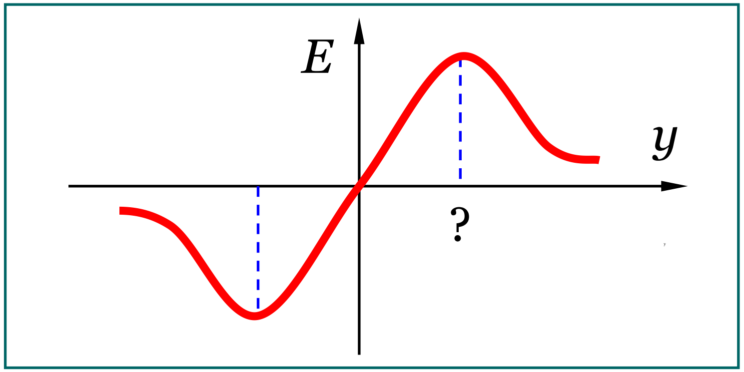
$1-3y^2/(y^2+a^2)=0$

$y = \pm\sqrt{2}a/2$

$E_y^{max} = \pm 2/3^{3/2}\, q/(4\pi\varepsilon_0 a^2)$

$= \pm 0.3849\, q/(4\pi\varepsilon_0 a^2)$

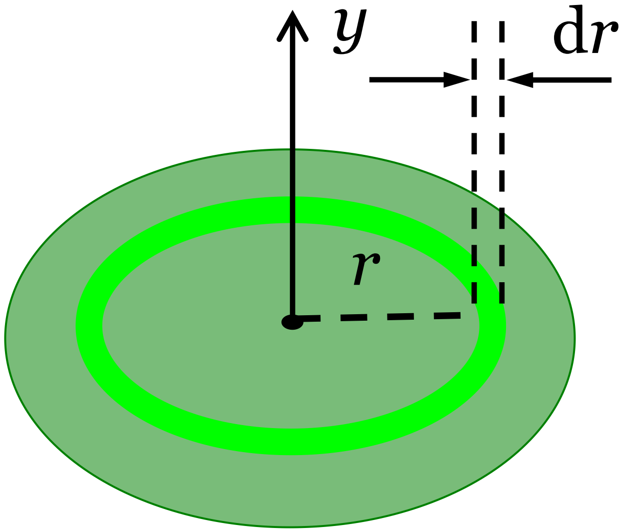




1.2 Electric field

Example 4: For a charged plate with surface density

$\sigma = q/\pi R^2$ ----> $dq = 2\pi r \sigma dr$



For any point at the central axis, $E_x = 0$,

which is protected by symmetry

$dE = y dq/[4\pi\varepsilon_0 (y^2+r^2)^{3/2}]$

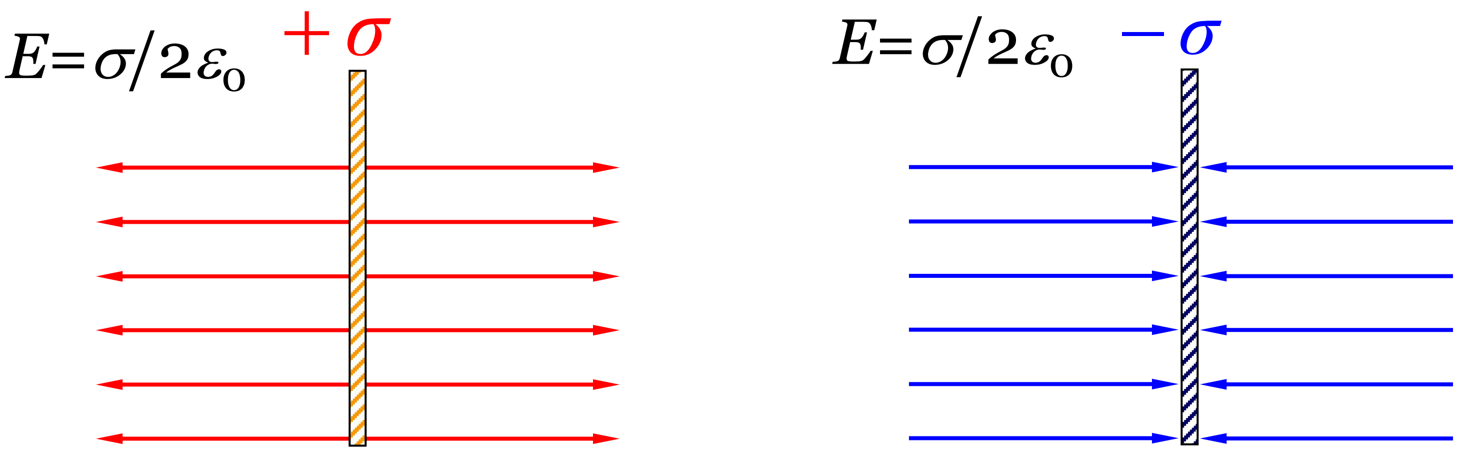
$E = \int 2\pi r y \sigma dr/[4\pi\varepsilon_0 (y^2+r^2)^{3/2}]$

$= \sigma/(2\varepsilon_0)\ [1 - y/(R^2+y^2)^{1/2}]$

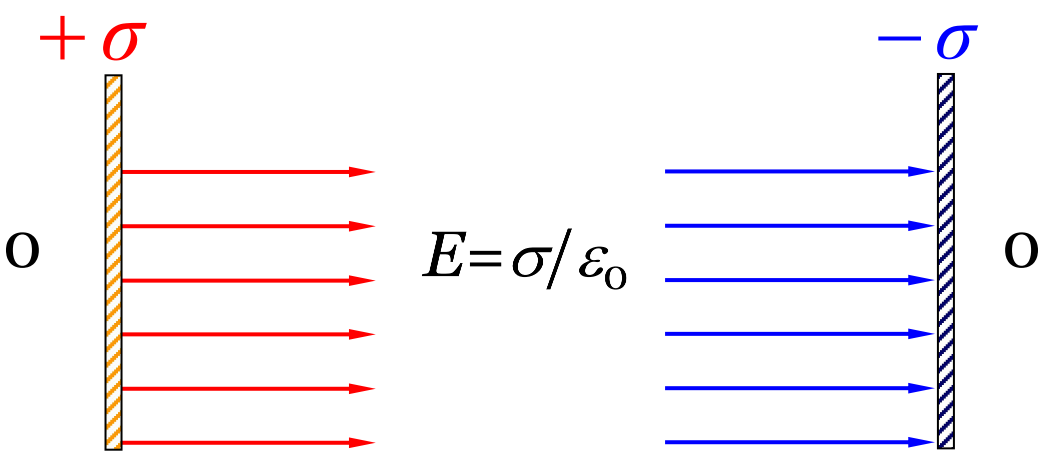
if $y \to \infty >> R$, $E = \sigma/(2\varepsilon_0) R^2/(2y^2) = q/(4\pi\varepsilon_0 y^2)$, back to the point charge limit

else if $R \to \infty >> y$, $E = \sigma/(2\varepsilon_0)$, which becomes a constant

1.2 Electric field



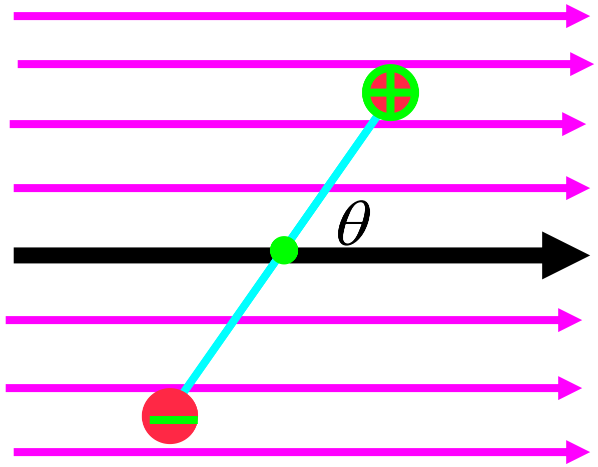
For two infinite large parallel plates with opposite charge, $E=\sigma/\varepsilon_0$ between plates but $E=0$ outside



Electrostatic screening effect

1.2 Electric field

Example 5: The force of dipole in electric field:



$f^+ = Eq$; $f^- = -Eq$

$f = f^+ + f^- = 0$

The torque of dipole in electric field:

$L^+ = f^+\, l/2 \sin\theta$; $L^- = -f^-\, l/2 \sin\theta$

$L = L^+ + L^- = Eql \sin\theta$

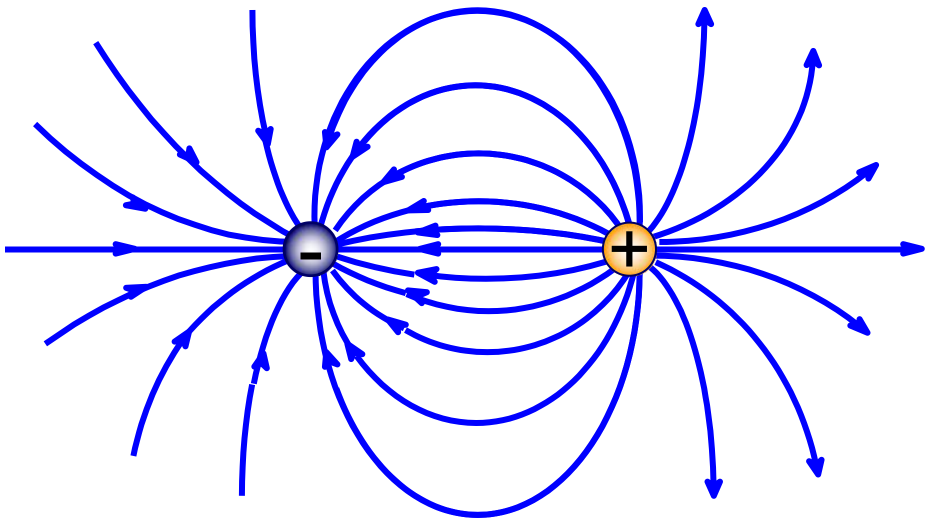
or expressed in the vector form: $\mathbf{L} = \mathbf{p} \times \mathbf{E}$

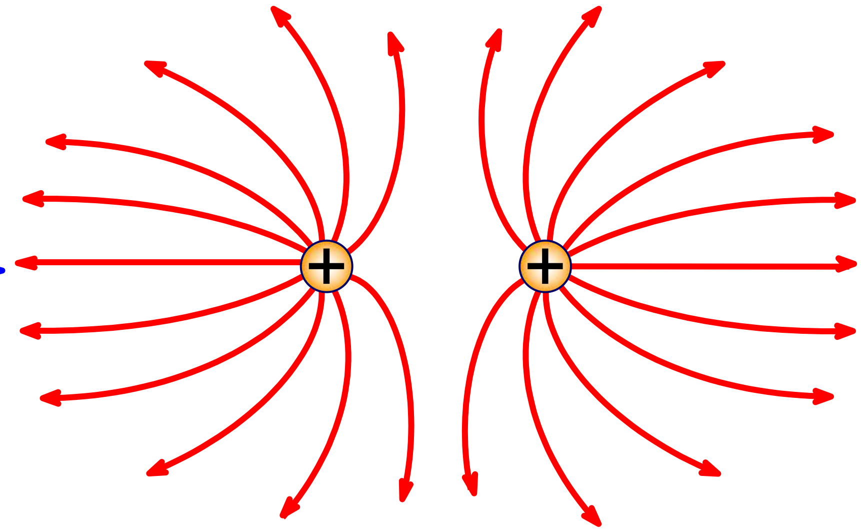
$\theta = \pi/2$ or $3\pi/2$, maximum L;

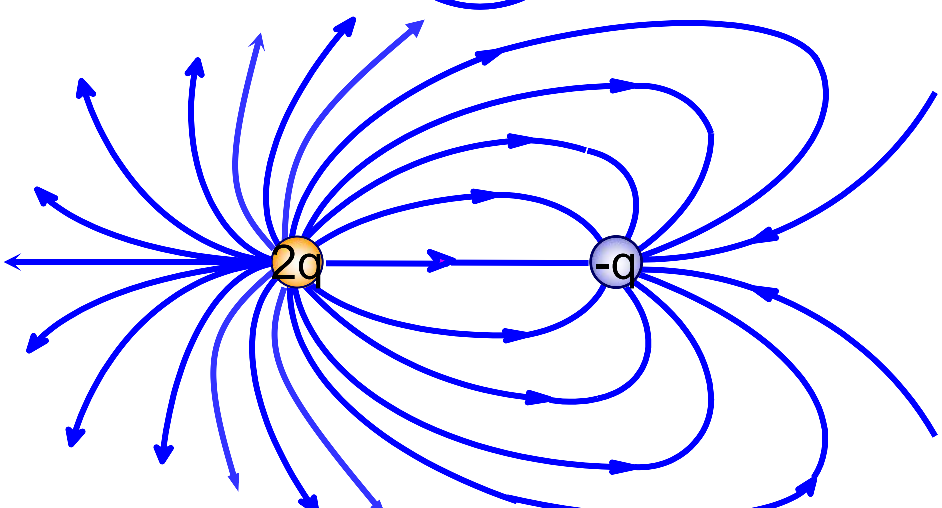
$\theta = 0$ or π, $L = 0$; $\theta = 0$ (ground state); $\theta = \pi$ (unstable state)

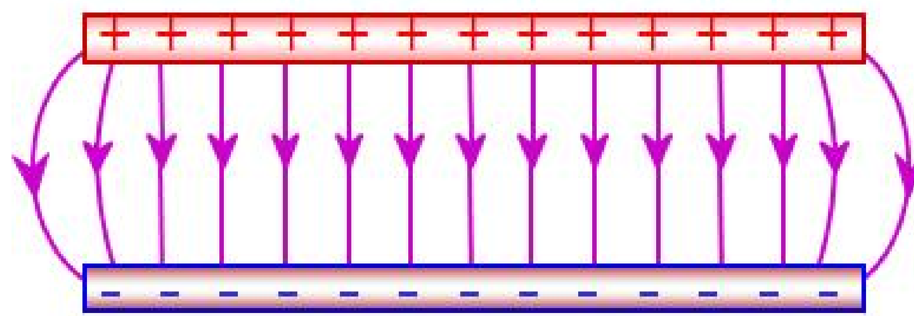
1.3 Gauss's law

Electric field lines are an excellent way of visualizing electric fields.







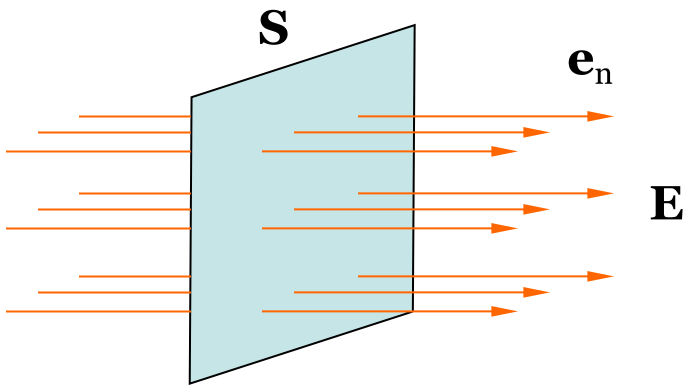


1.3 Gauss's law

Electric field flux Φ

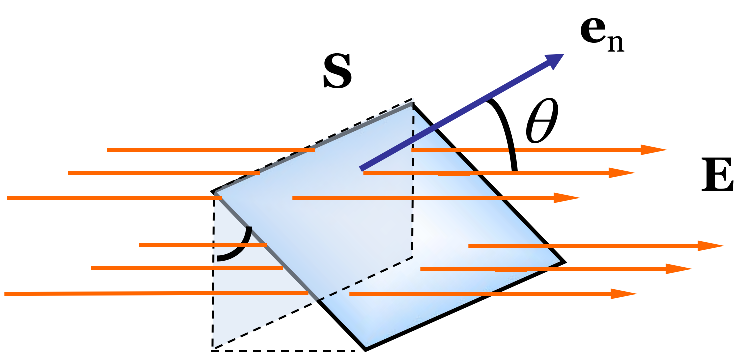
Uniform **E** perpendicular to an area **S**

$\Phi = \mathbf{E} \cdot \mathbf{S} = ES$



Uniform **E** with a canting angle θ to an area **S** ($\mathbf{e}_n$)

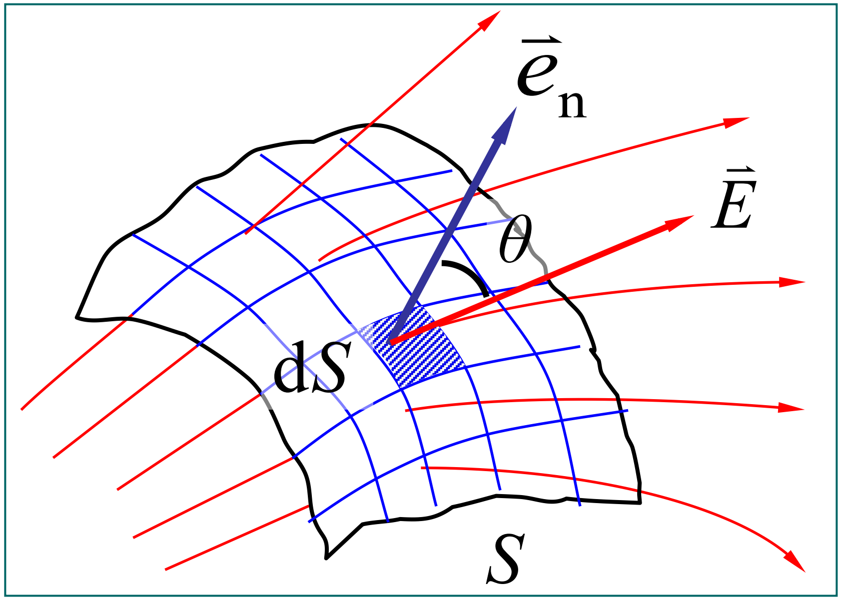
$\Phi = \mathbf{E} \cdot \mathbf{S} = ES\cos\theta$



1.3 Gauss's law

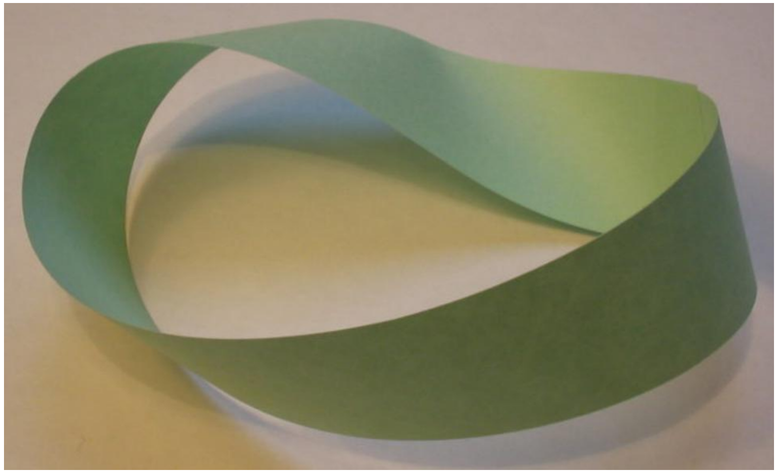
For arbitrary field and arbitary surface

$\Phi = \int d\Phi = \int_S \mathbf{E} \cdot d\mathbf{S}$



Note: the sign of Φ depends on the choice of normal direction

Question: how many normal directions for a surface? front & back



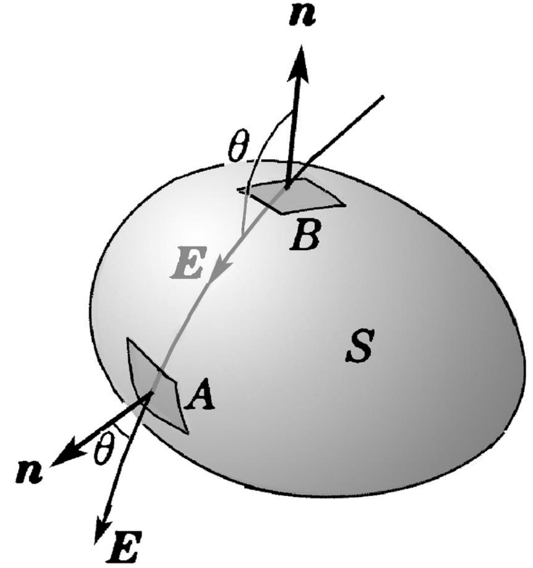
Möbius strip

1.3 Gauss's law

For close surface

$$\Phi = \oint d\Phi = \oint \mathbf{E}\cdot d\mathbf{S}$$

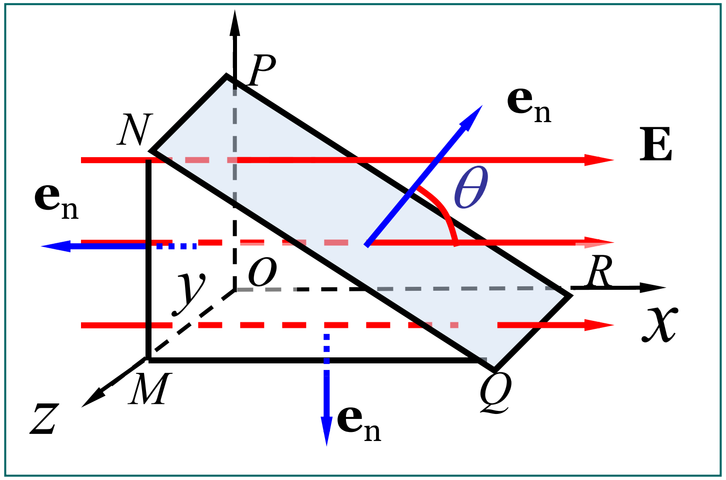
The normal direction points outside.



outflow **E**: $0 < \theta < 90^o$
influx **E**: $90^o < \theta < 180^o$

Example 1: a prism in uniform **E**

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5$$
$$= \Phi_1 + \Phi_2 = -ES + ES = 0$$



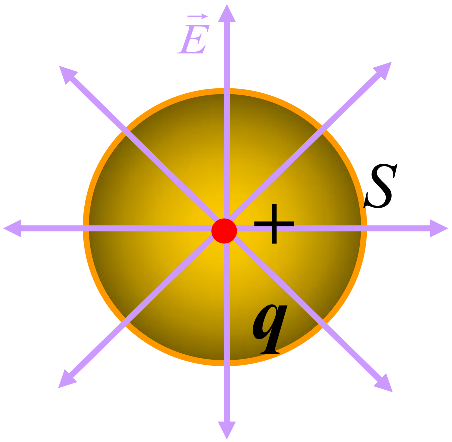
1.3 Gauss's law

$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

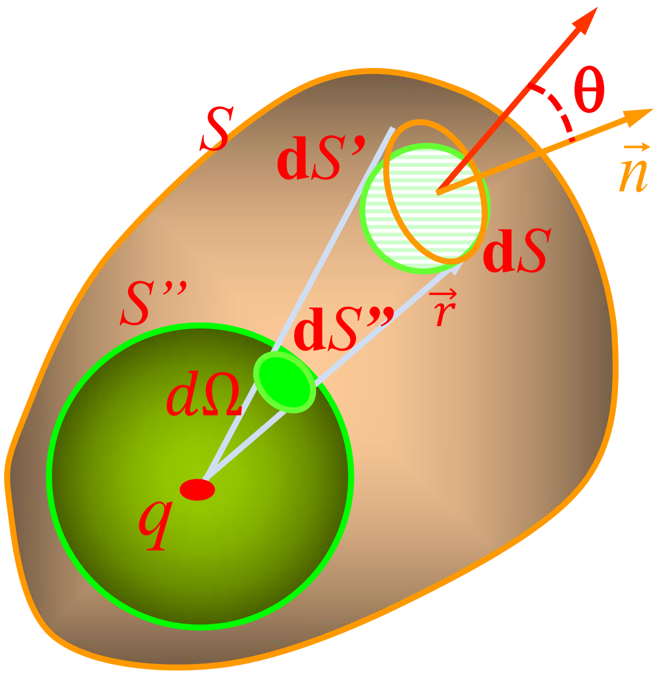
$$\Phi_e = \oiint_S \mathrm{d}\Phi_e = \oiint_S \frac{q}{4\pi\varepsilon_0 r^2}\mathrm{dS}$$

For a sphere with radius *r* and point charge at the center

$$\Phi = q/(4\pi\varepsilon_0 r^2)\, 4\pi r^2 = q/\varepsilon_0$$



For arbitrary close surface, the concept of solid angle helps, which leads an invariant conlusion.



1.3 Gauss's law

With source/drain

$\Phi = q/\varepsilon_0$

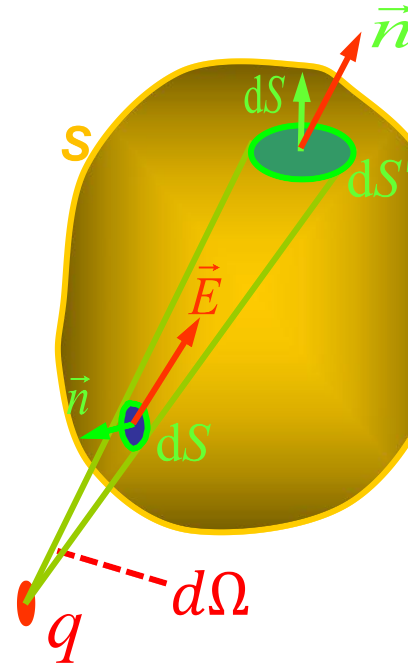
Without source/drain

$\Phi = 0$

More thinking on the Coulomb's law

Johann Carl Friedrich Gauss 1777-1855

For any close surface, for any shape of charge source



$$\Phi_e = \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} q_i^{in}$$

Gauss's flux theorem

1.3 Gauss's law

$\oiint \mathbf{E}\cdot d\mathbf{S} = \sum_V q_i/\varepsilon_o$ ⇒ $\oiint \mathbf{E}\cdot d\mathbf{S} = 1/\varepsilon_o \iiint \rho_e dV$

V is the volume enclosed by the surface S

Gauss's divergence theorem

$$\oiint \mathbf{E}\cdot d\mathbf{S} = \iiint \nabla\cdot\mathbf{E}\, dV$$

Operator nabla (del): $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

$$\iiint \nabla\cdot\mathbf{E}\, dV = 1/\varepsilon_o \iiint \rho_e dV$$

For any volume ⇒ $\nabla\cdot\mathbf{E} = \rho_e/\varepsilon_o$

Coulomb's law ⇔ Gauss's law

1.3 Gauss's law

Some mathematic operators derived from nabla

$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

$\nabla \cdot \mathbf{E} = \partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z$ Divergence: div·**E**

The divergence of electric field is in proportional to the charge density

$\nabla V = (\partial V/\partial x, \partial V/\partial y, \partial V/\partial z)$ Gradient: grad *V*

The gradient of electric potential is in proportional to the electric field

1.3 Gauss's law

Some mathematic operators derived from nabla

$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

$\nabla^2 = \nabla \cdot \nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

Laplacian operator: div grad, del squared

The del squared of electric potential is in proportional to the charge density

$\nabla \times \mathbf{A} = (\partial A_z/\partial y - \partial A_y/\partial z, \partial A_x/\partial z - \partial A_z/\partial x, \partial A_y/\partial x - \partial A_x/\partial y)$

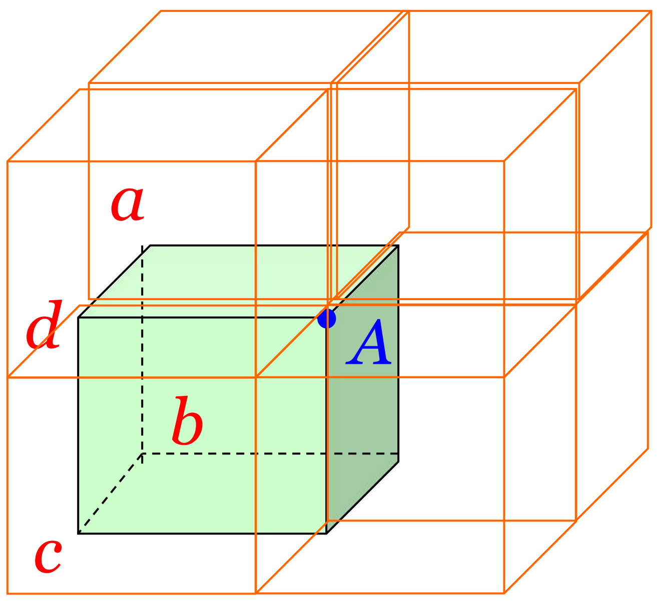
Curl: del cross

Additional homework 2:

To prove $\nabla \times \mathbf{E} = 0$ for electrostatic field

1.3 Gauss's law

Example: what's the electric field flux through the square area "abcd", once a point charge q is placed at the corner A?

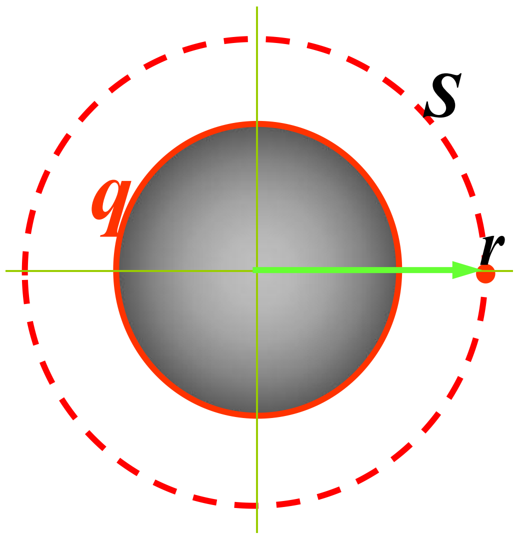


outside surface:
$2 \times 2 \times 6 = 24$

$\Phi = 1/24\ q/\varepsilon_0$

Symmetry: a powerful rule

1.3 Gauss's law



Example 2: what's the electric field at P for a uniformly charged spherical surface?

Symmetry: spherical

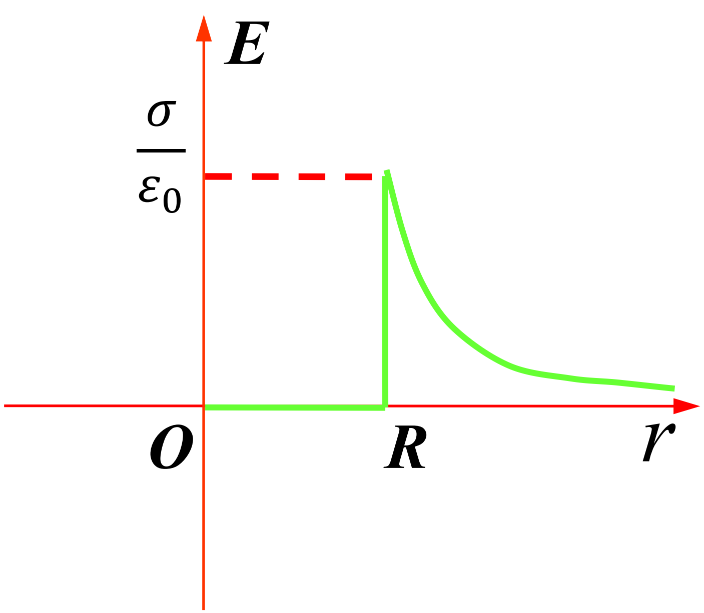
$\mathbf{E}(r, \theta, \varphi) = \mathbf{E}(r) \,||\, \mathbf{r}$,

outside: $4\pi r^2 E = q/\varepsilon_0$

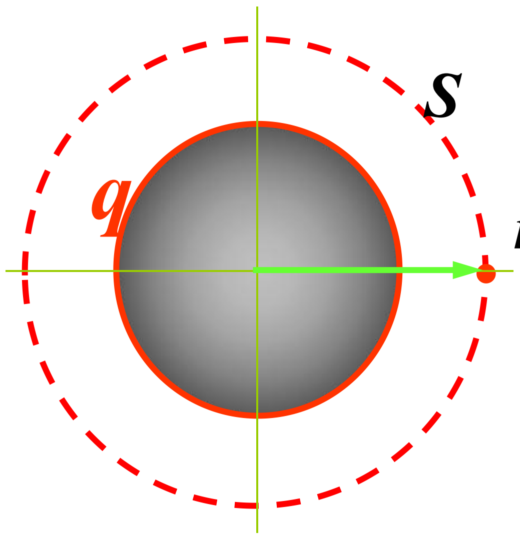
---> $E = q/(4\pi\varepsilon_0 r^2) = \sigma/\varepsilon_0$

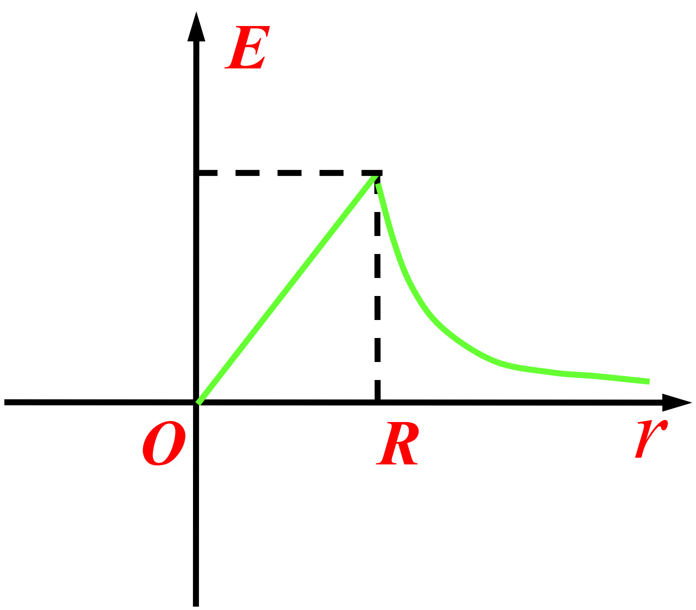
The same as the case of point charge

inside: $4\pi r^2 E = 0$ ---> $E = 0$



1.3 Gauss's law





Example 3: what's the electric field at r for a uniformly charged ball?

Symmetry: spherical

$\mathbf{E}(r, \theta, \varphi) = \mathbf{E}(r) \parallel \boldsymbol{r}$,

outside: $4\pi r^2 E = q/\varepsilon_0$

---> $E = q/(4\pi\varepsilon_0 r^2)$

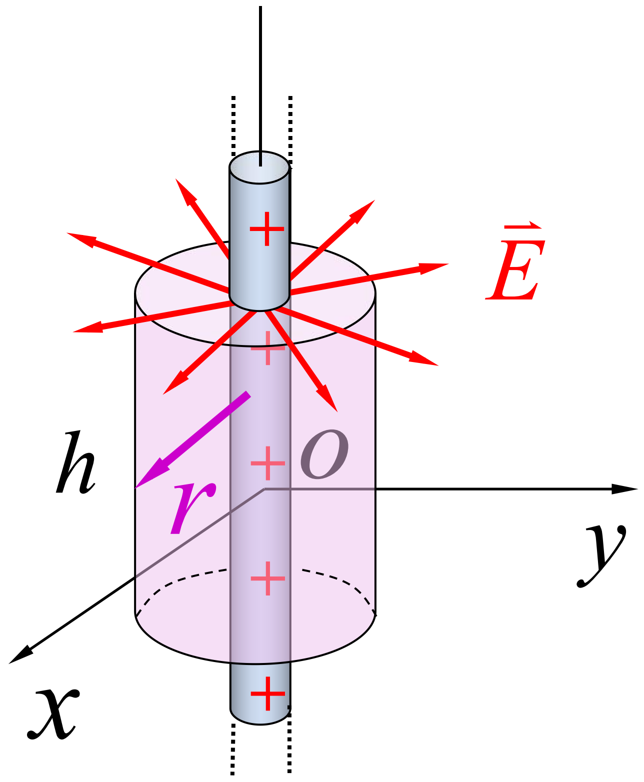
The same as the case of point charge

inside: $4\pi r^2 E = q/\varepsilon_0 \, (r/R)^3$

---> $E = qr/(4\pi\varepsilon_0 R^3)$

1.3 Gauss's law

Example 4: what's the electric field at r for a uniformly charged infinite long rod?



Symmetry: axial

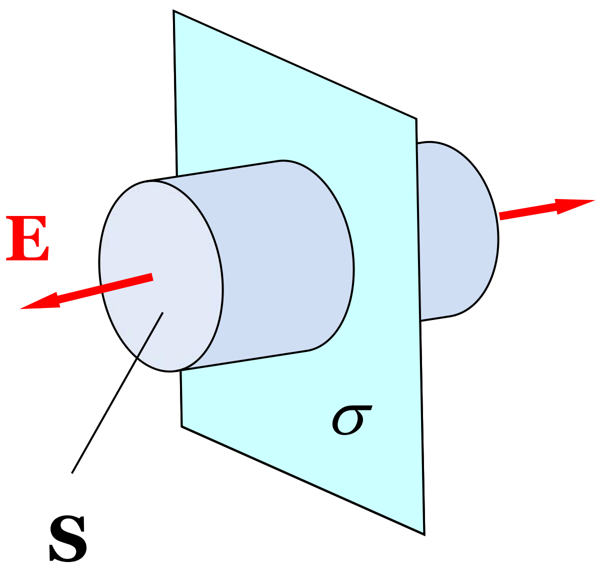
$\mathbf{E}(r, \theta, \varphi) = \mathbf{E}(r) \,||\, \mathbf{r}$, $E_z=0$

outside: $2\pi rhE=\eta h/\varepsilon_0$

---> $E=\eta/(2\pi\varepsilon_0 r)$

1.3 Gauss's law

Example 5: what's the electric field at r for a uniformly charged infinite flat sheet?

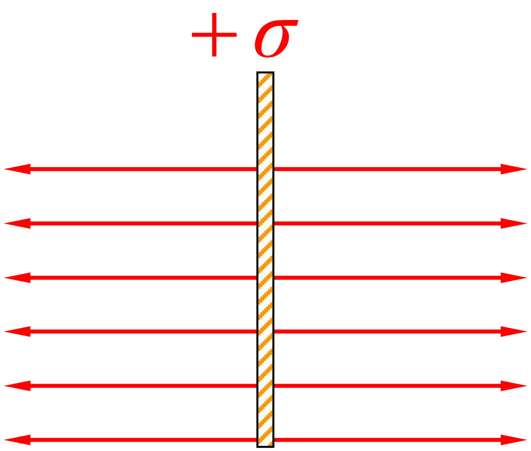


Symmetry: translation & mirror

$\mathbf{E}(\mathbf{r})\ ||\ \mathbf{n}$, $E_x=0$, $E_y=0$

$2ES=q/\varepsilon_0=\sigma S/\varepsilon_0$

---> $E=\sigma/(2\varepsilon_0)$

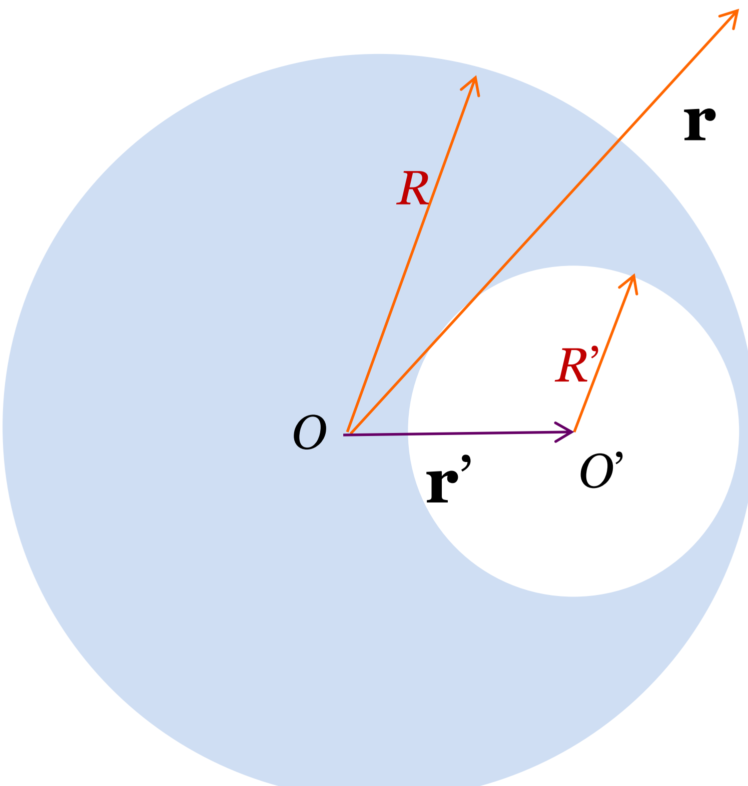


$E=\sigma/2\varepsilon_0$ according to aforementioned intergration

1.3 Gauss's law

Example 6: what's the outside electric field at **r** for a uniformly charged ball (charge density ρ) with cavity?

By filling this cavity to restore the symmetry



$q_O = 4\pi\rho R^3/3 \qquad q_{O'} = 4\pi\rho R'^3/3$

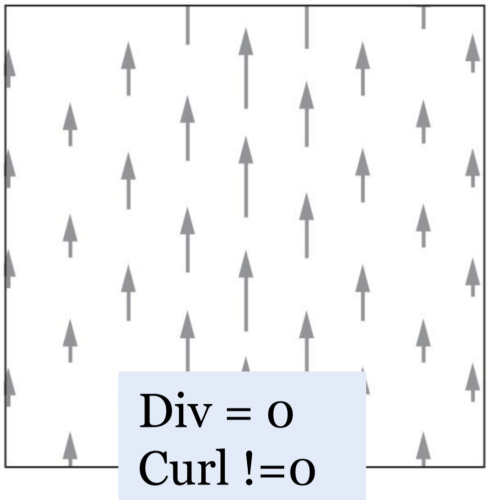
$\mathbf{E}_O = q_O\mathbf{r}/(4\pi\varepsilon_o r^3) = \rho\mathbf{r}R^3/(3\varepsilon_o r^3)$

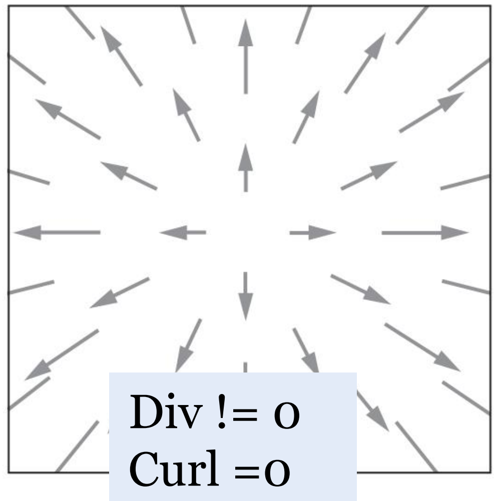
$\mathbf{E}_{O'} = q(\mathbf{r}-\mathbf{r}')/[4\pi\varepsilon_o(r-r')^3] = \rho(\mathbf{r}-\mathbf{r}')R'^3/[3\varepsilon_o(r-r')^3]$

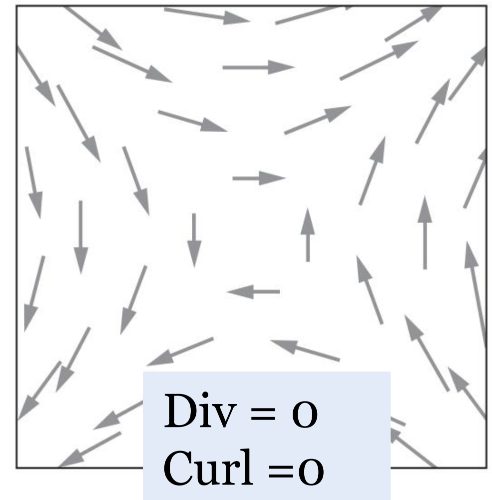
$\mathbf{E} = \mathbf{E}_O - \mathbf{E}_{O'} = \rho/(3\varepsilon_o)[\mathbf{r}R^3/r^3 - (\mathbf{r}-\mathbf{r}')R'^3/(r-r')^3]$

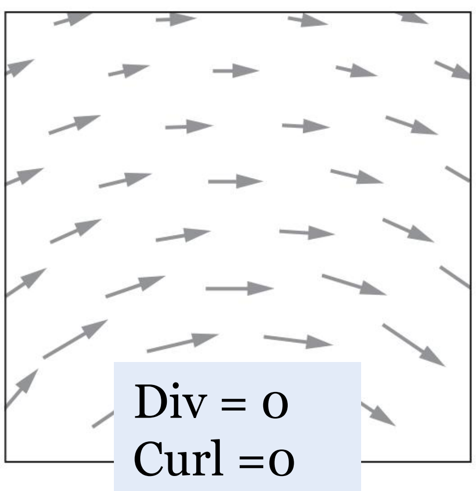
1.3 Gauss's law

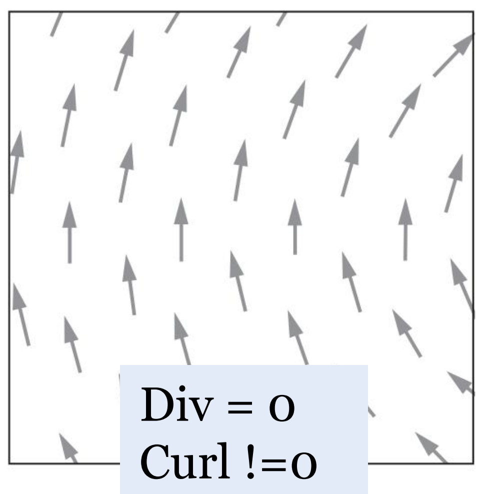
Which one(s) can be real electrostatic field?

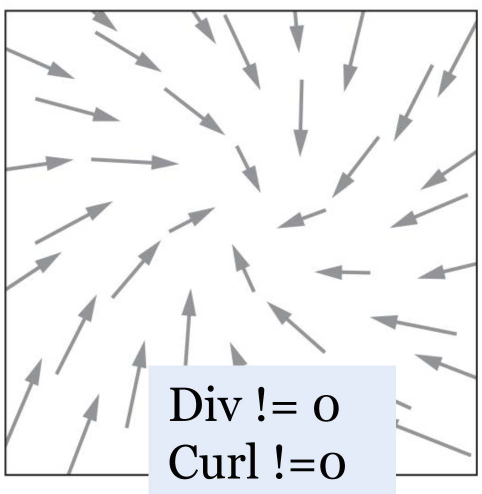






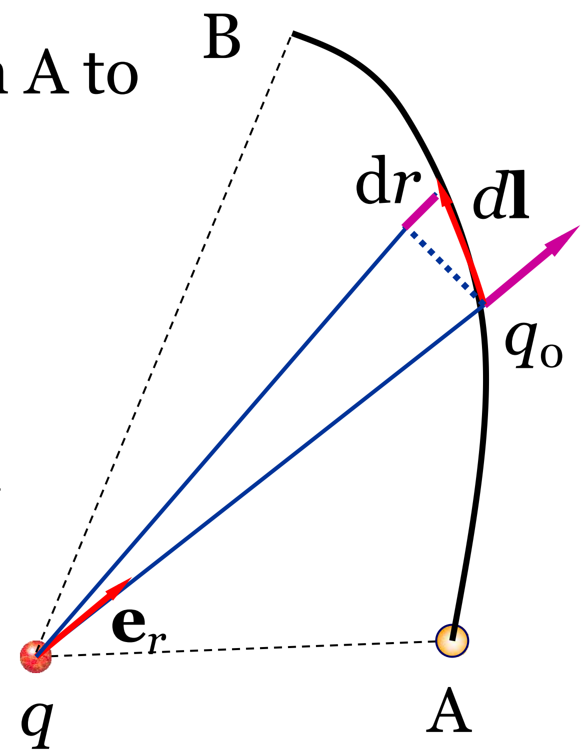






1.4 Electric potential

In an electrostatic field generated by a point charge q, a probe charge q_o moves from A to B. The work is:



$$dW = \mathbf{F}\cdot d\mathbf{l} = q\mathbf{E}\cdot d\mathbf{l} = qq_o/(4\pi\varepsilon_o r^2)\ \mathbf{e}_r\cdot d\mathbf{l}$$

$$\mathbf{e}_r\cdot d\mathbf{l} = \mathrm{d}r$$

$$W=\int_A^B dW = qq_o/(4\pi\varepsilon_o)\int dr/r^2$$

$$= qq_o/(4\pi\varepsilon_o)\ (1/r_A-1/r_B)$$

1.4 Electric potential

More than one point charge:

$$W=\int_A^B dW = q_o\int_A^B \mathbf{E}\cdot d\mathbf{l} = q_o\int_A^B (\mathbf{E}_1+\mathbf{E}_2+.....+\mathbf{E}_n)\cdot d\mathbf{l}$$

$$=q_o\int_A^B \mathbf{E}_1\cdot d\mathbf{l}+q_o\int_A^B \mathbf{E}_2\cdot d\mathbf{l}+.....+q_o\int_A^B \mathbf{E}_n\cdot d\mathbf{l}$$

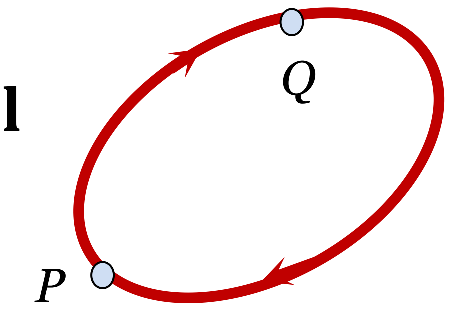
- *W* is independent on the path ---->
- The electrostatic field **E** is a conservative field
- An **E** field generated by point charge(s) is a conservative field??
- Another conservative field: gravitational field
- nonconservative field: ??

1.4 Electric potential

The line integral $\mathbf{E}\cdot d\mathbf{l}$ around any closed path in an electrostatic field is zero

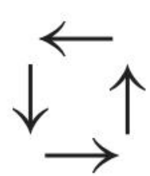
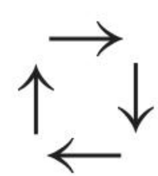
$$W = q_0 \oint_l \mathbf{E}\cdot d\mathbf{l} = q_0 \int_{P\,(L_1)}^{Q} \mathbf{E}\cdot d\mathbf{l} + q_0 \int_{Q\,(L_2)}^{P} \mathbf{E}\cdot d\mathbf{l}$$

$$= q_0 \int_{P\,(L_1)}^{Q} \mathbf{E}\cdot d\mathbf{l} - q_0 \int_{P\,(L_2)}^{Q} \mathbf{E}\cdot d\mathbf{l} = 0$$



Stokes' theorem $\oint \mathbf{A}\cdot d\mathbf{l} = \iint (\nabla\times\mathbf{A})\cdot d\mathbf{S}$

$\Rightarrow$ $(\nabla\times\mathbf{E})=0$ The curl of electrostatic field is zero

nonzero curl  or 

1.4 Electric potential

$U_{AB} = W_{AB}/q = \int_A^B \mathbf{E}\cdot d\mathbf{l} = U(A) - U(B)$

$U \sim -\int \mathbf{E}\cdot d\mathbf{l}$ unit: Volt;   1 V= 1 J/C

1 eV= 1.6×10^{-19} J

keV, MeV, GeV, TeV, meV, μeV

Note: U_{AB} is an absolute value, but

$U(A)$ is a relative value, which

needs a reference point.

Frequently used refereneces:

infinite far point or ground

Alessandro
Giuseppe Antonio
Anastasio Volta
1745-1827

The inventor of battery
(voltaic pile)

1.4 Electric potential

Voltage: potential difference

AA battery:
1.2-1.5 V

power bank ~5 V

For human safety <36 V

Home AC power

220-240 V (China and most)

110-120 V (USA, JP, ...)

High voltage power: 10 kV-1 MV

1.4 Electric potential

Example 1: potential of a point charge

$\mathbf{E}=q/(4\pi\varepsilon_0 r^2)\mathbf{e}_\mathrm{r}$

$U(\mathbf{r})=\int_r^\infty \mathbf{E}\cdot d\mathbf{l}=\int_r^\infty q\,dr/(4\pi\varepsilon_0 r^2)=q/(4\pi\varepsilon_0 r)-q/(4\pi\varepsilon_0 \infty)$

$U(\infty)=q/(4\pi\varepsilon_0\infty)=0$ $\quad\quad$ $U(\mathbf{r})=q/(4\pi\varepsilon_0 r)$

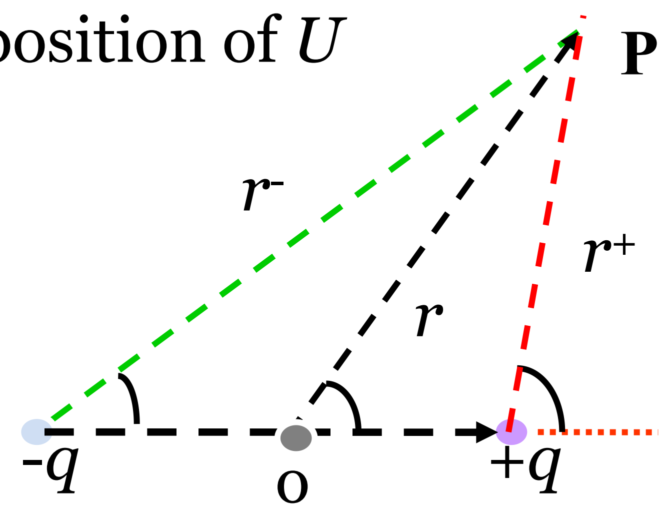
Example 2: potential of a dipole

Superposition of **E** ----> Superposition of *U*

$U(\mathbf{r})=U^+(\mathbf{r})+U^-(\mathbf{r})$

$=q/(4\pi\varepsilon_0 r^+)-q/(4\pi\varepsilon_0 r^-)$

$=q/(4\pi\varepsilon_0)\,(1/r^+-1/r^-)$



1.4 Electric potential

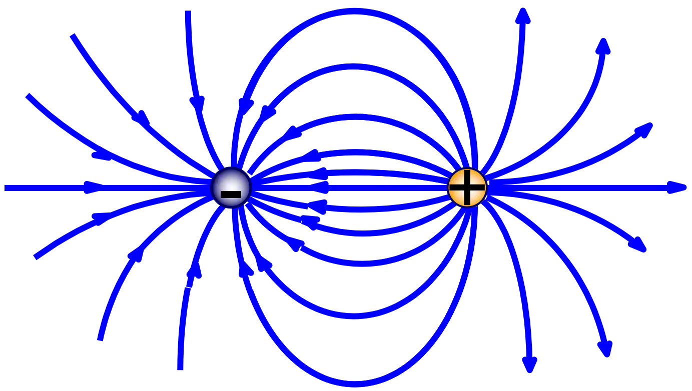
$U(\mathbf{r})=q/(4\pi\varepsilon_0)\,(1/r^+-1/r^-)$

Case 1: the verticle middle plane $r^+=r^-$

$U(x=0)=0$, independent of y or z (because $\mathbf{E}||\mathbf{x}$)

Case 2: the axial line $y=z=0$

2.1 $x>l/2$ $r^+=x-l/2$; $r^-=x+l/2$



$U(x)=q/(4\pi\varepsilon_0)\,(1/r^+-1/r^-)$

$=q/(4\pi\varepsilon_0)\,[l/(x^2-l^2/4)]$

2.2 $x<-l/2$ $r^+=-x+l/2$; $r^-=-x-l/2$

$U(x)=q/(4\pi\varepsilon_0)\,(1/r^+-1/r^-)$

$=q/(4\pi\varepsilon_0)\,[-l/(x^2-l^2/4)]$

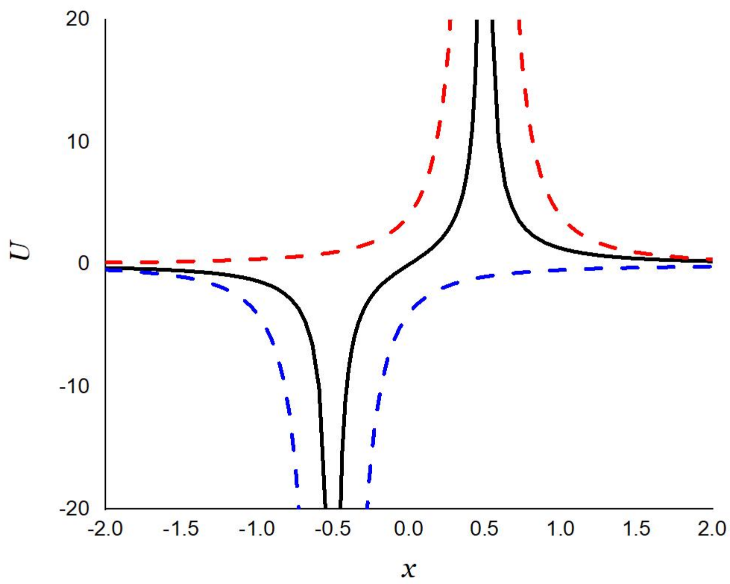
1.4 Electric potential

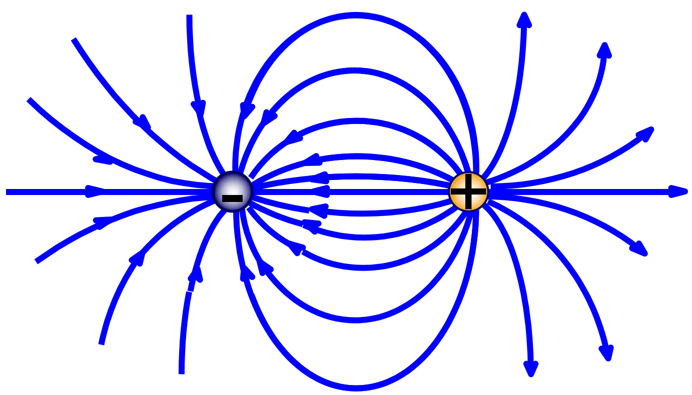
$U(\mathbf{r}) = q/(4\pi\varepsilon_0)\ (1/r^{+} - 1/r^{-})$

Case 2: the axial line $y = z = 0$

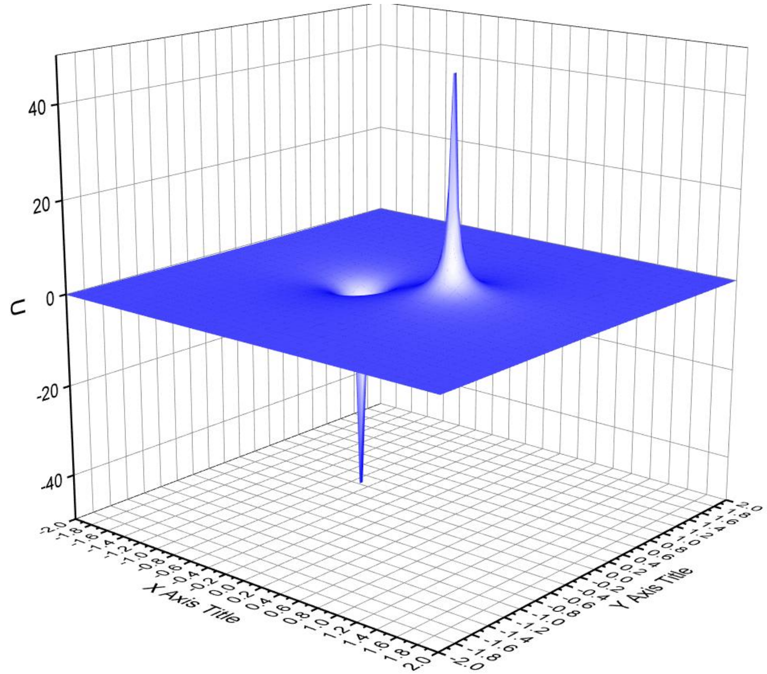
2.3 $-l/2 < x < l/2$ $\quad r^{+} = -x + l/2; \quad r^{-} = x + l/2$

$U(x) = q/(4\pi\varepsilon_0)\ (1/r^{+} - 1/r^{-}) = q/(4\pi\varepsilon_0)\ [2x/(-x^2 + l^2/4)]$





1.4 Electric potential



Case 3: other

$U(\mathbf{r})=q/(4\pi\varepsilon_0)\ (1/r^+-1/r^-)$

$\approx \mathbf{p}\cdot\mathbf{e}_r/(4\pi\varepsilon_0 r^2)$

Example 3: potential of a charged sphere ($q=4\pi\sigma R^2$)

$\mathbf{E}=q/(4\pi\varepsilon_0 r^2)\mathbf{e}_r \quad (r>R) \quad \mathbf{E}=0\ (r<=R)$

$U(\mathbf{r})=\int_r^\infty \mathbf{E}\cdot d\mathbf{l}=\int_r^\infty q\,dr/(4\pi\varepsilon_0 r^2)$

$=q/(4\pi\varepsilon_0 r)=\sigma R^2/(\varepsilon_0 r) \qquad (r>R)$

$U(\mathbf{r})=\int_r^R \mathbf{E}\cdot d\mathbf{l}+U(R)=\sigma R/\varepsilon_0 \qquad (r<=R)$

1.4 Electric potential

Example 4: potential of a uniformly charged ball

$(q=4\pi\rho R^3/3)$

$\mathbf{E}=q/(4\pi\varepsilon_0 r^2)\mathbf{e}_r \quad (r>R) \qquad \mathbf{E}=qr/(4\pi\varepsilon_0 R^3)\mathbf{e}_r \quad (r<=R)$

$U(\mathbf{r})=\int_r^\infty \mathbf{E}\cdot d\mathbf{l}=q/(4\pi\varepsilon_0 r)$

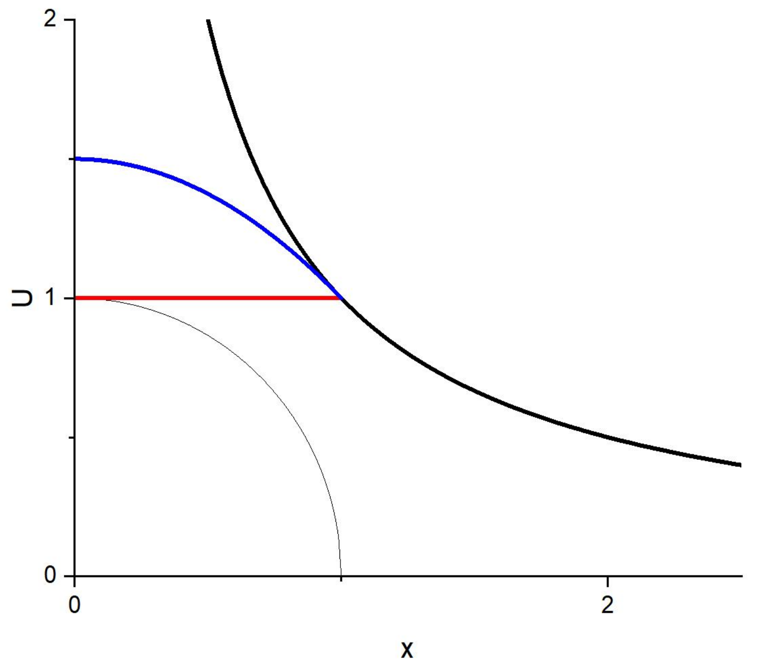
$=\rho R^3/(3\varepsilon_0 r) \qquad (r>R)$

$U(\mathbf{r})=\int_r^R \mathbf{E}\cdot d\mathbf{l}+U(R)$

$=q/(4\pi\varepsilon_0 R^3)\,(R^2-r^2)/2+q/(4\pi\varepsilon_0 R)$

$=q(3R^2-r^2)/(8\pi\varepsilon_0 R^3)$

$=\rho(3R^2-r^2)/(6\varepsilon_0) \qquad (r<=R)$



1.4 Electric potential

Example 5: potential of a uniformly charged infinite long rod (R=0) $\mathbf{E}=\eta/(2\pi\varepsilon_0 x)\,\mathbf{e}_x$

$U(x)=\int_x^\infty \mathbf{E}\cdot d\mathbf{l}=\int_x^\infty \eta dx/(2\pi\varepsilon_0 x)=\eta/(2\pi\varepsilon_0)[\ln(\infty)-\ln(x)]$

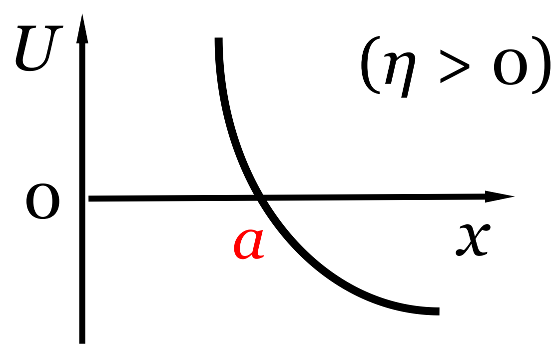
$U(\infty)=\eta/(2\pi\varepsilon_0)\ln(\infty)$ is infinite (divergent)

Reference point: ∞? 0?

$U(x)=\int_0^x \mathbf{E}\cdot d\mathbf{l}=\int_0^x \eta dx/(2\pi\varepsilon_0 x)=\eta/(2\pi\varepsilon_0)[\ln(x)-\ln(0)]$

$U(0)=\eta/(2\pi\varepsilon_0)\ln(0)$ is also infinite (divergent)

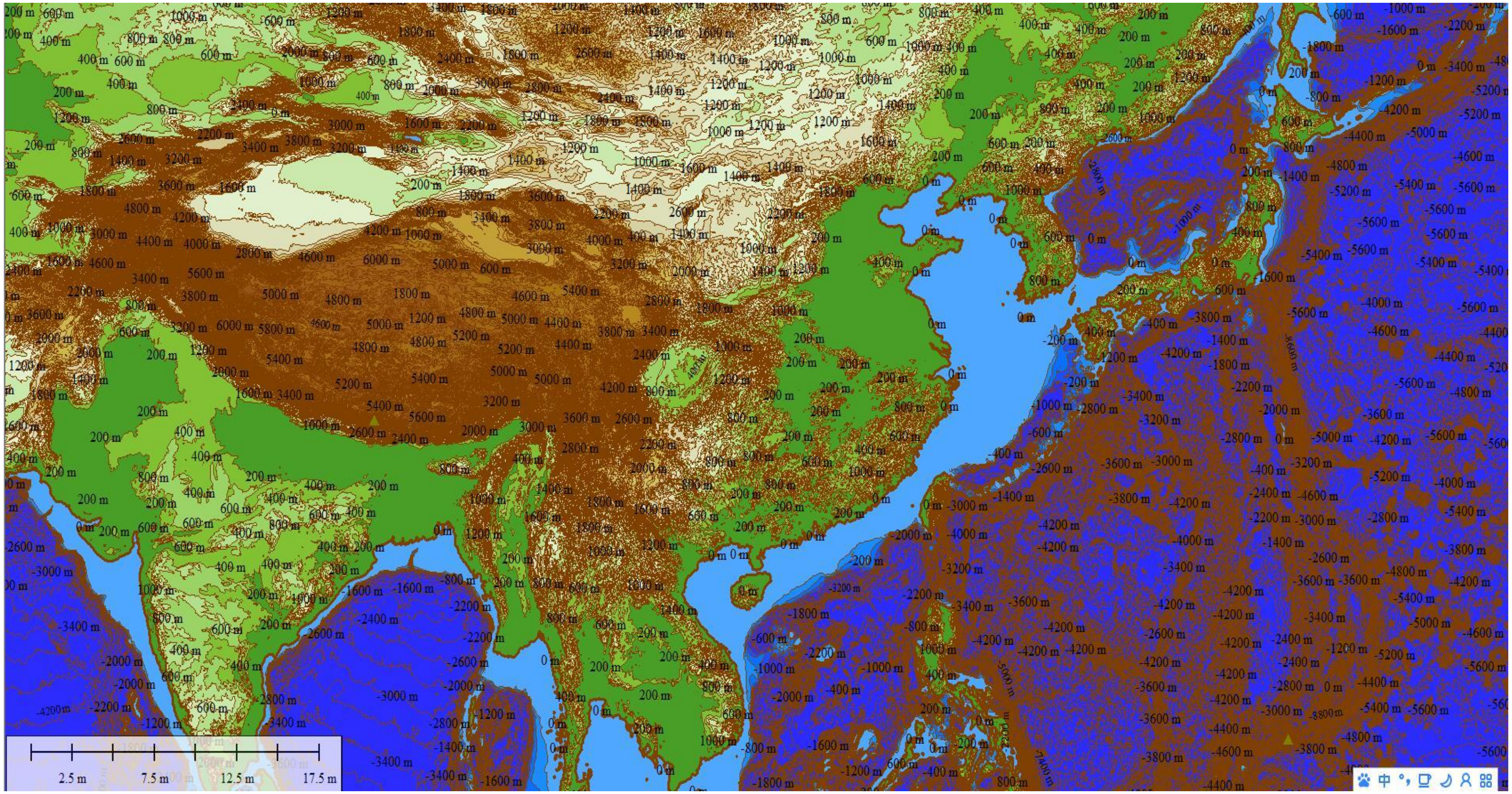
$U(x)=\int_x^a \mathbf{E}\cdot d\mathbf{l}=-\eta/(2\pi\varepsilon_0)[\ln(x)-\ln(a)]$



1.4 Electric potential

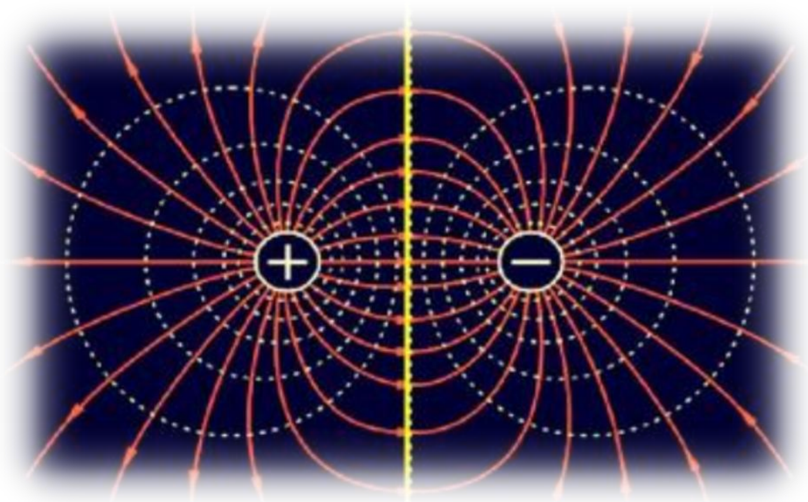
Isopotential surface (contour map).

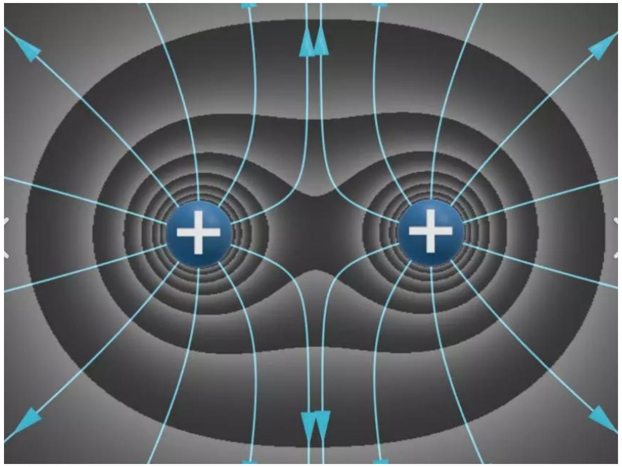
E is always perpendicular to this surface everywhere .

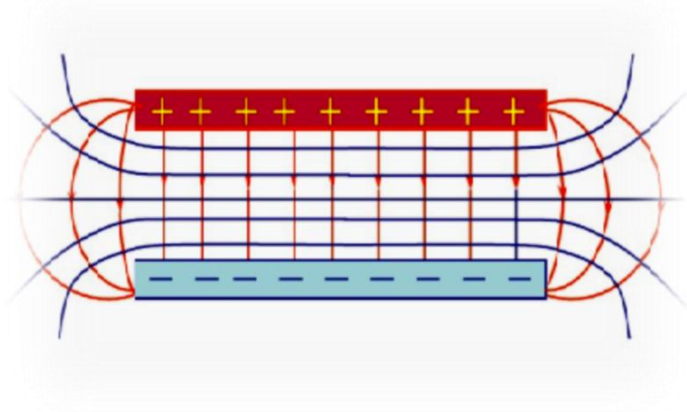


1.4 Electric potential

Isopotential surface.







The inverse action of integral:

differentiation

$-\partial U/\partial r = E_r$

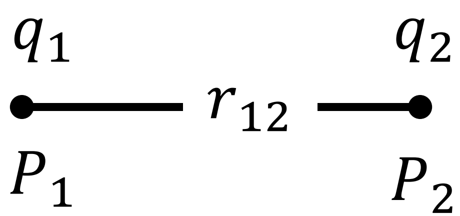
$-\nabla U = \mathbf{E}$

E: not necessary to be continuous,

U: must be continuous

1.5 Electrostatic energy

Two point charges



$\mathbf{F}_{12}=q_2\mathbf{E}_1=-\mathbf{F}_{21}=-q_1\mathbf{E}_2$

By moving q_2 from infinite far to P_2, the energy gain of this charge pair is:

$$A = -\int_{\infty}^{P2} \mathbf{F}_{12}\cdot d\mathbf{l} =-q_2\int_{\infty}^{P2} \mathbf{E}_1\cdot d\mathbf{l}=q_2 U_{12}$$

$$U_{12} = U_1(P_2) = -\int_{\infty}^{P2} \mathbf{E}_1\cdot d\mathbf{l}=q_1/(4\pi\varepsilon_0 r_{12})$$

$$A = q_1 q_2/(4\pi\varepsilon_0 r_{12})$$

Or by moving q_1 from infinite far to P_1, the energy gain of this charge pair is also A

1.5 Electrostatic energy

If there are more point charges, let's move charge one by one from inifinite far to certain points:

$$\begin{cases} A_1=0 \\ A_2=q_2U_{12} \\ A_3=q_3U_{13}+q_3U_{23} \\ \cdots\ \cdots\ \cdots\ \cdots \\ A_n=q_n(U_{1n}+U_{2n}+\ldots+U_{n-1,n}) \end{cases}$$

$$A_i = q_i \sum_{j=1}^{i-1} U_{ji}$$

$$U_{ji} = U_j(P_i) = -\int_{\infty}^{P_i} \mathbf{E}_j \cdot d\mathbf{l} = q_j/(4\pi\varepsilon_0 r_{ij})$$

1.5 Electrostatic energy

$$A = A_1+A_2+\ldots+A_n = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} q_i \sum_{j=1}^{i-1} U_{ji}$$

$$= 1/(4\pi\varepsilon_0) \sum_{i=1}^{n} \sum_{j=1}^{i-1} q_i q_j / r_{ij}$$

$$= 1/(8\pi\varepsilon_0) \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j / r_{ij} \quad (i \neq j)$$

$$= 1/2 \sum_{i=1}^{n} q_i U_i$$

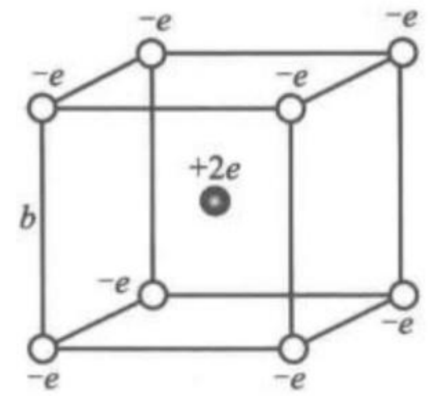
$$W=A = 1/2 \sum_{i=1}^{n} q_i U_i \quad (i \neq j)$$

W: interaction energy

e.g. 100 point charges, 100×99/2=4950 pairs. U_i is the potential generated at i by other 99 charges.

1.5 Electrostatic energy

Example 1: electrostatic energy of this cube.



1. the 2nd neighbors (-e & -e)

$$W_1 = 12 \times e^2/(4\pi\varepsilon_0 b)$$

2. the nearest neighbors (-e & 2e)

$$W_2 = 8 \times -2e^2/(4\pi\varepsilon_0 \sqrt{3}b/2) = -32/\sqrt{3} \times e^2/(4\pi\varepsilon_0 b)$$

3. the 3rd neighbors (-e & -e)

$$W_3 = 12 \times e^2/(4\pi\varepsilon_0 \sqrt{2}b) = 12/\sqrt{2} \times e^2/(4\pi\varepsilon_0 b)$$

4. the 4th neighbors (-e & -e)

$$W_4 = 4 \times e^2/(4\pi\varepsilon_0 \sqrt{3}b) = 4/\sqrt{3} \times e^2/(4\pi\varepsilon_0 b)$$

$$W = W_1 + W_2 + W_3 + W_4 = e^2/(4\pi\varepsilon_0 b)\,(12 + 6\sqrt{2} - 28\sqrt{3}/3)$$

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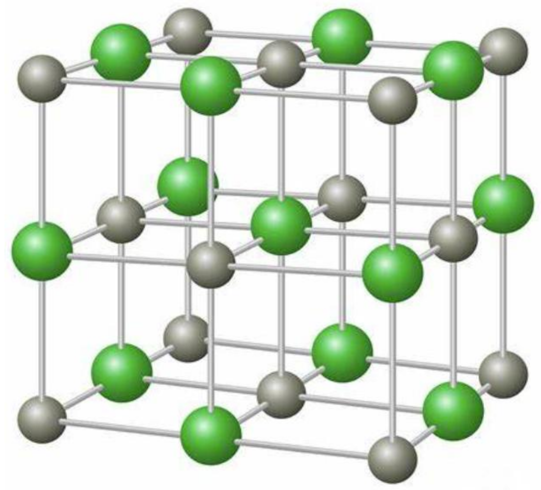
Example 2: electrostatic energy of NaCl crystal



Identity principle

$N(N$-1) pairs ---> N sites/2

1. the nearest neighbors (e & $-e$)

$$W_1^+=6\times -e^2/(4\pi\varepsilon_0 a)$$

2. the 2nd neighbors (e & e)

$$W_2^+=12\times e^2/(4\pi\varepsilon_0\sqrt{2}a)$$

3. the 3rd neighbors (e & $-e$)

$$W_3^+=8\times -e^2/(4\pi\varepsilon_0\sqrt{3}a)$$

$$W^+ = W_1^+ + W_2^+ + W_3^+ + \ldots.$$

$$=e^2/(4\pi\varepsilon_0 a)\,(-6+6\sqrt{2}-8/\sqrt{3}+\ldots)$$

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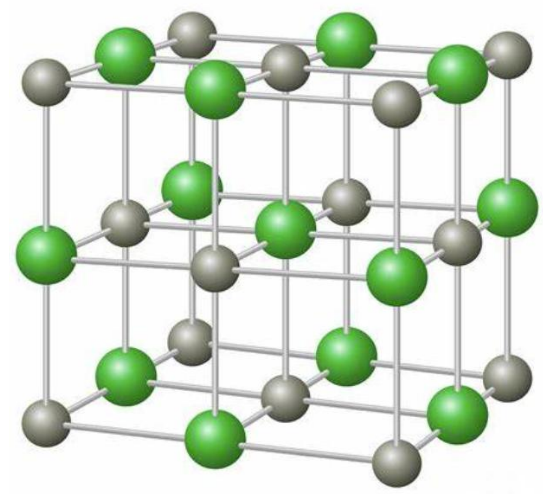
Example 2: electrostatic energy of NaCl crystal

$W^+ = e^2/(4\pi\varepsilon_0 a)\,(-6+6\sqrt{2}-8/\sqrt{3}+\ldots)$



an infinite series

$W^+ = -0.8738\, e^2/(4\pi\varepsilon_0 a) < 0$

$W^- = W^+$

$W_{NaCl} = (W^- + W^+)/2 = W^+$

$= -0.8738 \times 1.6 \times 10^{-19}/(4\pi \times 8.85 \times 10^{-12} \times 2.815 \times 10^{-10})$

$= -4.465$ eV cohesive energy?

Madelung constant: 1.748 for NaCl

Different Madelung constants for other crystals

1.5 Electrostatic energy

For continuous distribution of charge, interaction energy ---> self energy

$W = 1/2 \sum_{i=1}^{n} q_i U_i$ ⇒ $W = 1/2 \iiint \rho U dV$ (3D)

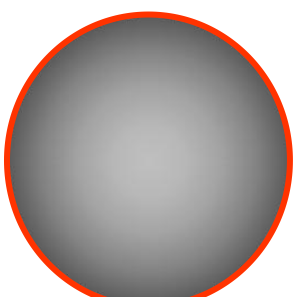
$W = 1/2 \int \eta U dl$ (1D) $W = 1/2 \iint \sigma U dS$ (2D)

Example 3: a uniformly charged sphere surface

$U(\mathbf{r}) = \sigma R^2/(\varepsilon_0 r)$ ($r > R$)

$U(\mathbf{r}) = \sigma R/\varepsilon_0$ ($r <= R$)

3D or 2D?



3D: $\rho = \sigma\,\delta_{r=R}$ Kronecker delta function

2D: $W_s = 1/2 \iint \sigma U dS = 2\pi\sigma^2 R^3/\varepsilon_0 = q^2/(8\pi\varepsilon_0 R)$

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Example 4: a uniformly charged ball

$U(\mathbf{r})=q/(4\pi\varepsilon_0 r)=\rho R^3/(3\varepsilon_0 r)$ $(r>R)$

$U(\mathbf{r})=q(3R^2-r^2)/(8\pi\varepsilon_0 R^3)=\rho(3R^2-r^2)/(6\varepsilon_0)$ $(r<=R)$

Only need to integrate within the ball

$W_b=1/2 \iiint \rho U dV= \rho^2/(12\varepsilon_0)\iiint (3R^2-r^2)dV$

$= \rho^2/(12\varepsilon_0) (4\pi R^5 -4\pi\int r^4 dr)=\rho^2/(12\varepsilon_0) (4\pi R^5 -4\pi R^5/5)$

$=4\pi R^5\rho^2/(15\varepsilon_0) =3q^2/(20\pi\varepsilon_0 R)$

For comparison, the self energy of uniformly charged sphere

1.5 Electrostatic energy

Ball $3q^2/(20\pi\varepsilon_0 R)$ > Sphere $q^2/(8\pi\varepsilon_0 R)$

Charge will stay at the surface of a conducting ball

Example 5: radius of an electron

charge: $q=1.6\times10^{-19}$ C mass: $m=9.1\times10^{-31}$ kg

speed of light $c=3\times10^{8}$ m/s

Energy $W=mc^2=q^2/(8\pi\varepsilon_0 R)$ $R=1.4\times10^{-15}$ m=1.4 fm

Chapter 1: Homework

1. 8-1, 8-2, 8-4, 8-5, 8-6, 8-11, 8-14
2. Additional homework 1: comparison of electric field generated by H_2O and CO_2 molecules
3. Additional homework 2: to prove $\nabla \times \mathbf{E} = 0$ for electrostatic field

- Deadline: Friday, April 26