Research on the Behaviors of 1D and 2D Solitary Waves

Based on Soliton Solutions of KdV Equation and Sine-Gordon Equation

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Abstract

Solitary waves are used widely in scientific and practical applications due to their distinctive physical properties. Many researchers have been trying to find analytic descriptions of solitary waves to further study their characteristics. In this report, we use numerical methods to investigate the soliton solutions of 1D KdV equation and 2D sine-Gordon equation, and then describe the behaviors of solitary waves revealed by those solutions, including the formation, propagation, collision, superposition and evolution of solitary waves.

Keywords: Solitary waves; Soliton; KdV Equation; Sine-Gordon Equation; Finite-difference Method

1. Introduction

The concept of a solitary wave was first introduced by J. Scott Russell when he observed unusual water waves in 1834. He described this phenomenon in his report:

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon…

Since then, efforts have been paid to find equations that describe this unique phenomenon. It was not until Korteweg and de Vries provided an analytic description of solitary waves (Korteweg-de Vries Equation or KdV equation) in 1895 that solitary waves were widely accepted by academic communities and experienced a short time of prosperity. After about 60 years of little progress on research on solitary waves, in 1955, Enrico Fermi, John Pasta and Stan Ulam published a paper entitled *Studies of Nonlinear Problems* arousing researchers' interests in solitary waves again. In the paper, they introduced the concept of "soliton" to describe solitary waves that behave like particles. However, those two concepts have almost no difference today.

Nowadays, solitary waves have been playing increasingly significant roles in both scientific research and practical applications. Their distinctive properties still fascinate many researchers, and more of their applications are waiting to be discovered.

In this report, we are interested in two equations (KdV equation and sine-Gordon equation) that have soliton solutions. We solve these equations for their soliton solutions and discuss what properties of solitons these solutions can tell us.

2. Methods

2.1 1D KdV Equation:

$$\frac{\partial u(x,t)}{\partial t}+\varepsilon u(x,t)\frac{\partial u(x,t)}{\partial x}+\mu\frac{\partial^3 u(x,t)}{\partial x^3}=0 \quad (1)$$

Equation (1) above can be solved numerically by finite-difference method.

The time and space derivatives are given by the central-difference approximations:

$$\frac{\partial u(x,t)}{\partial t}\simeq\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta t},\frac{\partial u(x,t)}{\partial x}\simeq\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} \quad (2)$$

The term $\frac{\partial^3 u(x,t)}{\partial x^3}$ can be approximated by Taylor expansion:

$$u(x\pm\Delta x,t)\simeq u(x,t)\pm(\Delta x)\frac{\partial u}{\partial x}+\frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}\pm\frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} \quad (3)$$

We can take the average of three x values with the same t for u(x,t):

$$u(x,t)\simeq\frac{u_{i+1,j}+u_{i,j}+u_{i-1,j}}{3} \quad (4)$$

After substituting these approximations above, we can get the algorithm for KdV equation:

$$u_{i,j+1}\simeq u_{i,j-1}-\frac{\varepsilon}{3}\frac{\Delta t}{\Delta x}[u_{i+1,j}+u_{i,j}+u_{i-1,j}][u_{i+1,j}-u_{i-1,j}] -\mu\frac{\Delta t}{(\Delta x)^3}[u_{i+2,j}+2u_{i-1,j}-2u_{i+1,j}-u_{i-2,j}] \quad (5).$$

u(x,0) can be obtained by initial conditions, and we can apply forward-difference scheme to get u(x,1):

$$u_{i,1}=u_{i,0}-\frac{\varepsilon}{6}\frac{\Delta t}{\Delta x}[u_{i+1,0}+u_{i,0}+u_{i-1,0}][u_{i+1,0}-u_{i-1,0}] -\frac{\mu}{2}\frac{\Delta t}{(\Delta x)^3}[u_{i+2,0}+2u_{i-1,0}-2u_{i+1,0}-u_{i-2,0}] \quad (6).$$

For those undefined points, we assume that

$$u_{0,1}=1,\ u_{0,2}=1,$$

$$u_{N\max-1,1}=0,\ u_{N\max-1,2}=0 \quad (7).$$

2.2 2D Sine-Gordon Equation (SGE):

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=\sin u \quad (8)$$

Equation (8) can be solved by finite-difference method like what we have discussed in KdV equation.

The algorithm for SGE is:

$$u_{m.l}^{n+1}\simeq -u_{m.l}^{n-1}+\left(\frac{\Delta t}{\Delta x}\right)^2[u_{m+1.l}^n+u_{m-1.l}^n+u_{m.l+1}^n+u_{m.l-1}^n] -(\Delta t)^2\sin[\frac{1}{4}(u_{m+1.l}^n+u_{m-1.l}^n+u_{m.l+1}^n+u_{m.l-1}^n)] \quad (9).$$

We can take $\Delta t=\Delta x/\sqrt{2}$ to make the algorithm simpler and stable. By using this trick, u(x,y,1) can be obtained in a simpler form:

$$u_{m.l}^1\simeq\frac{1}{4}[u_{m+1.l}^0+u_{m-1.l}^0+u_{m.l+1}^0+u_{m.l-1}^0] -\frac{(\Delta t)^2}{2}\sin[\frac{1}{4}(u_{m+1.l}^0+u_{m-1.l}^0+u_{m.l+1}^0+u_{m.l-1}^0)] \quad (10)$$

We take the initial condition and boundary conditions as follows:

$$\frac{\partial u}{\partial t}(x,y,t=0)=0 \quad (11),$$

$$\frac{\partial u}{\partial x}(-x_0,y,t)=\frac{\partial u}{\partial x}(x_0,y,t)=0 \quad (12)$$

$$\frac{\partial u}{\partial y}(x,-y_0,t)=\frac{\partial u}{\partial y}(x,y_0,t)=0 \quad (13).$$

2.3 Algorithm

i. Define a 2D array u(x,t) for KdV equation or a 3D array u(x,y,t) for SGE.

ii. Initialize u(x,t=0) or u(x,y,t=0) with initial conditions.

iii. Use the formula (6) for KdV or (10) for SGE and apply boundary conditions (7) for KdV or (12) for SGE to get u(x,t=1) or u(x,y,t=1).

iv. Use the formula (5) for KdV or (9) for SGE to get the third time step u(x,t=2) or u(x,y,t=2).

v. For the sake of saving computers' memory, at the end of every iteration, we make u(x,t=0) = u(x,t=1) and u(x,t=1) = u(x,t=2), or u(x,y,t=0) = u(x,y,t=1) and u(x,y,t=1) = u(x,y,t=2).

vi. Repeat step (iv) and (v) to continue the propagation.

(A C++ program for solving KdV equation and SGE is provided in attachment.)

3. Results and Discussions

3.1 1D Solitons

3.1.1 Solitons in the propagation of a two-level waveform

The initial conditions is:

$$\frac{1}{2}\left[1-\tanh\left(\frac{x-25}{5}\right)\right]$$

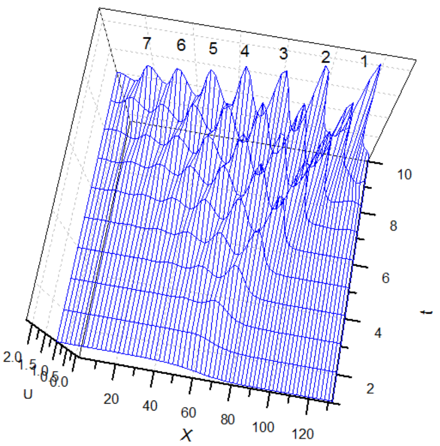


Fig. 1. One two-level waveform breaks up into seven solitons (label 1-7).

Figure 1 shows that a single two-level waveform breaks up into seven solitons as time increases. This figure is a simulation of tsunamis: when a sudden change exists in the level of an ocean floor and propagates over a long distance with no attenuation and dispersion, tsunamis tend to form.

We calculate velocities of the solitons with three different amplitudes (A>B>C) by tracing their positions as time increases and linearly fit those points (see Fig. 2). Table 1 shows the results of fitting:

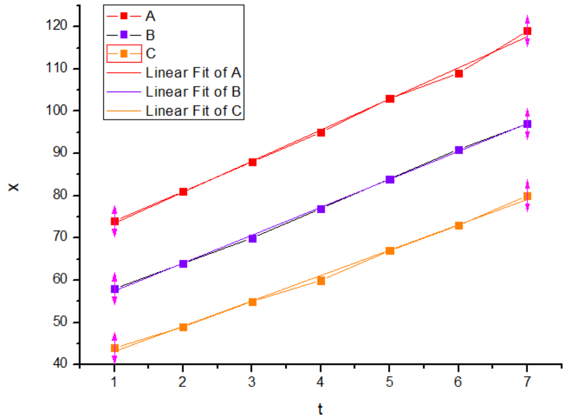


Fig. 2. Different velocities of solitons with different amplitudes.

Soliton	R-Square	Slope
A	0.9967	7.357
B	0.9989	6.607
C	0.9966	6.000

Table 1. Linear fit of different solitons' x-t curve.

Since the R-squares are very close to 1, which means straight lines fit the data well, the slopes can represent solitons' velocities (dx/dt). We can draw a conclusion from figure 2 and table 1 that a soliton with larger amplitude travels faster than a smaller one, and moreover, the velocities of every soliton is invariant during propagation. Errors here may be caused by parameter selections and data processing.

3.1.2 Collision of two solitons

The initial condition is:

$$0.8\sinh^2(\frac{3x}{12}-3)+0.3\sinh^2(\frac{4.5x}{26}-4.5)$$

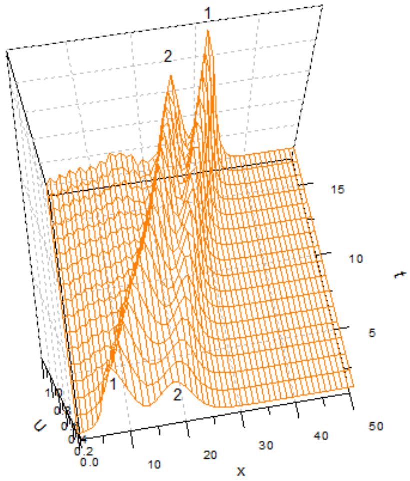


Fig. 3. Collision of two solitons

Figure 3 shows that two solitons (label 1 and 2) collide with each other during propagation due to the difference in velocities between the taller (faster) soliton (label 1) and the shorter (slower) soliton (label 2). We also find that after colliding, these two solitons retain their initial velocities and waveforms. Moreover, since the amplitudes of the two solitons are both positive, if the superposition of them is linear, the amplitudes should be larger than that of either of the two solitons when they are superposed. However, we can see from figure 3 that this is not the case, and thus, the superposition of the two solitons is nonlinear.

3.2 2D Solitons (quasi-solitons)

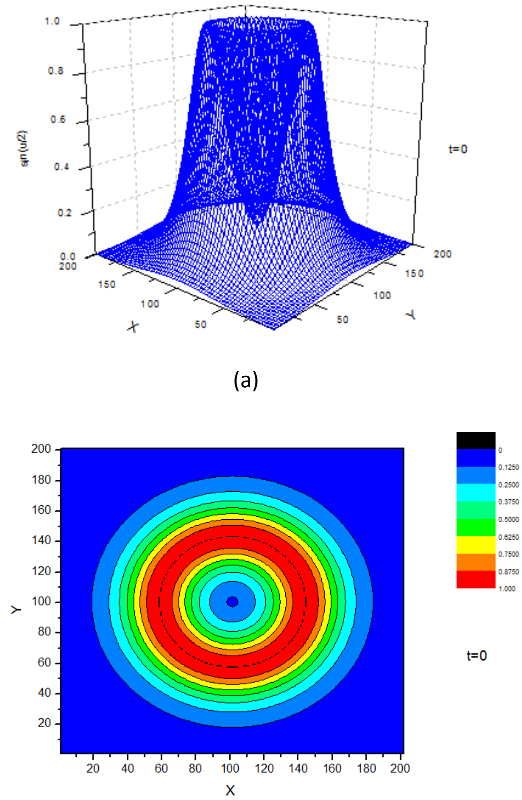
We use OriginPro to show the intuitive appearance of the soliton at different times by wire frame and draw the contours with amplitudes for observing the behaviors of solitons more directly.

3.2.1 Circular ring solitons

Figure 4.1 to 4.5 show the evolution of a circular ring soliton. The initial condition is:

$$u(x,y,0)=4\arctan(e^{3-\sqrt{x^2+y^2}}) \qquad -7\le x,y\le 7$$

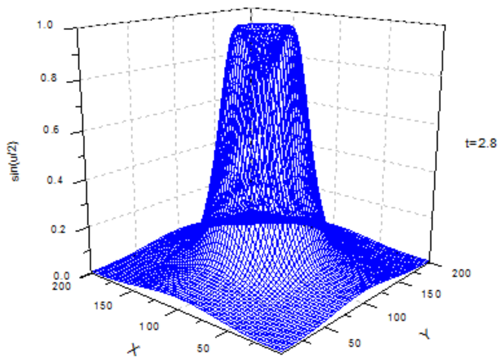
$$u'(x,y,0)=0 \qquad -7\le x,y\le 7$$



(a)

(b)

Fig. 4.1. Circular ring solitons at t=0.



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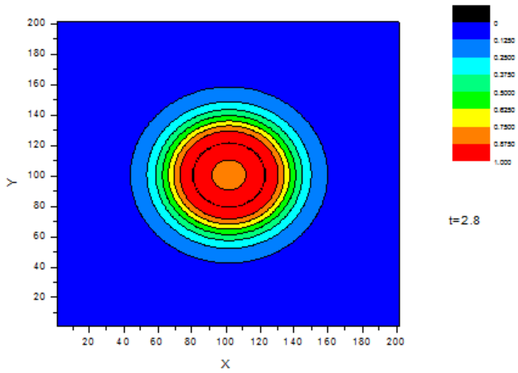

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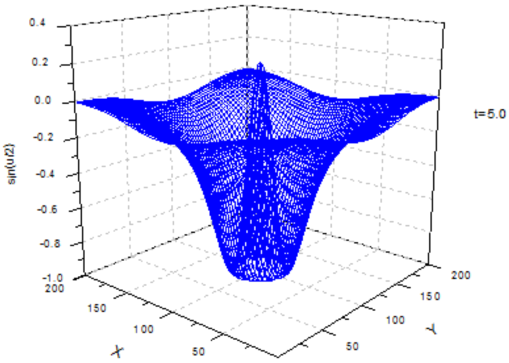
Fig. 4.2. Circular ring solitons at t=2.8.



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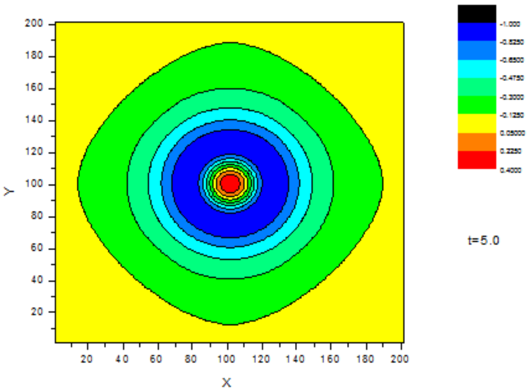

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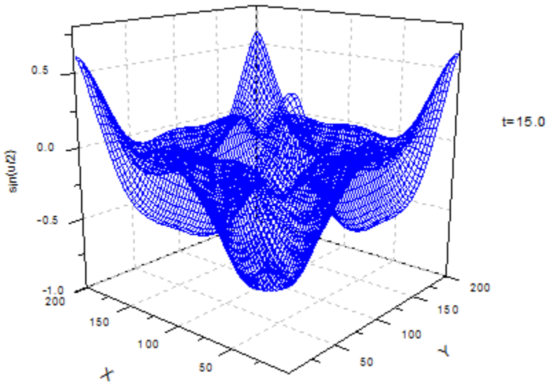
Fig. 4.3. Circular ring solitons at t=5.0.



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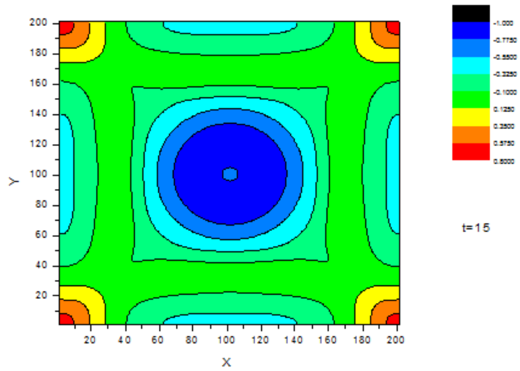

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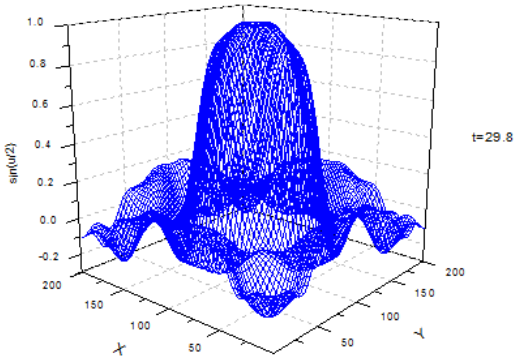
Fig. 4.4. Circular ring solitons at t=15.0.



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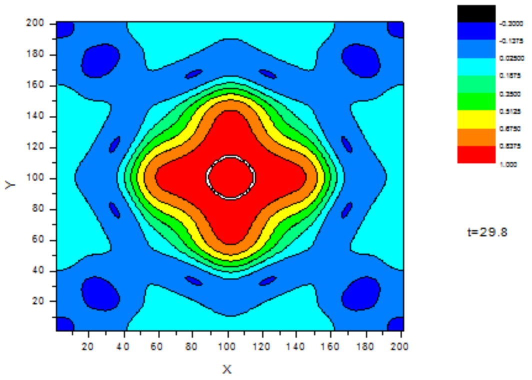

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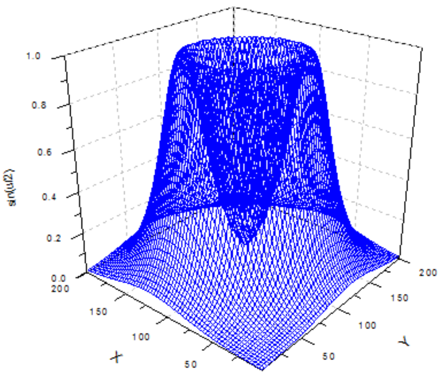
Fig. 4.5. Circular ring solitons at t=29.8.

Figure 5.1 to 5.4 show the evolution of an elliptical ring soliton (an animation is also provided in attachment). The initial condition is:

$$u(x,y,0)=4\arctan(e^{3-\sqrt{(x-y)^2/3+(x+y)^2/2}})\qquad -7\le x,y\le 7$$

$$u'(x,y,0)=0\qquad -7\le x,y\le 7$$



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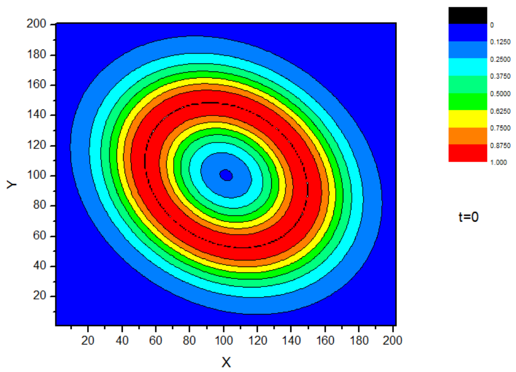

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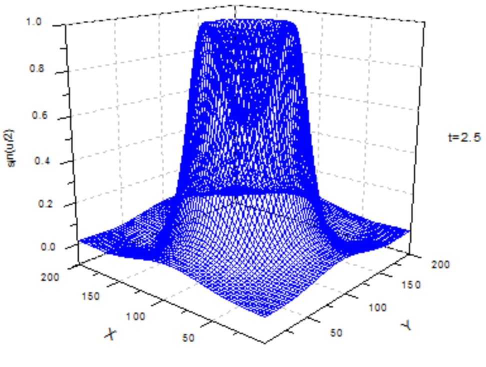
Fig. 5.1. Elliptical ring solitons at t=0.



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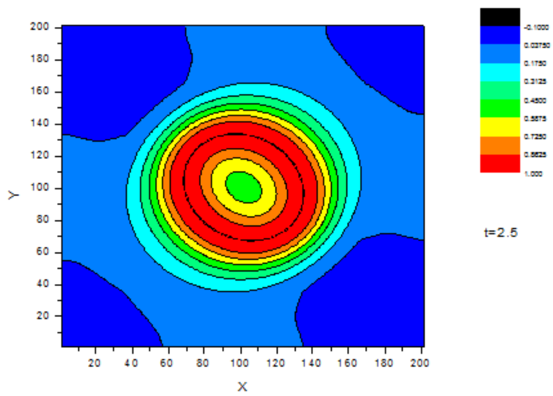

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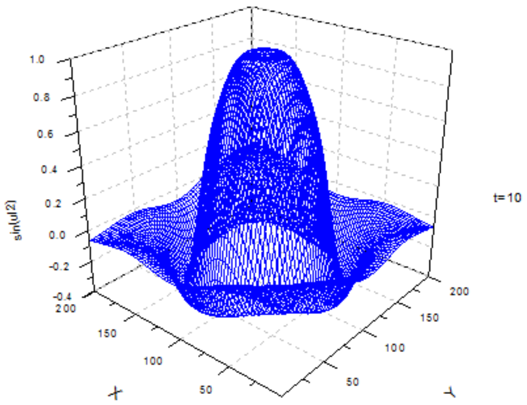
Fig. 5.2. Elliptical ring solitons at t=2.5.



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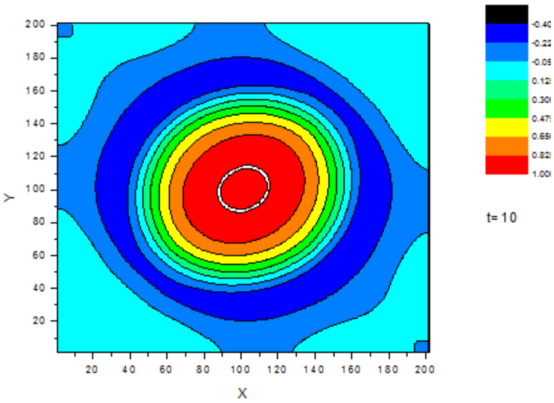

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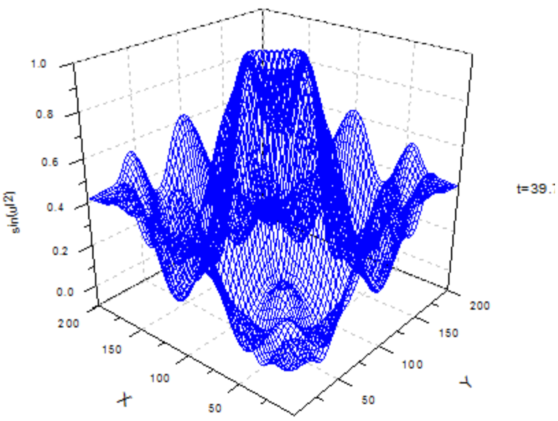
Fig. 5.3. Elliptical ring solitons at t=10.



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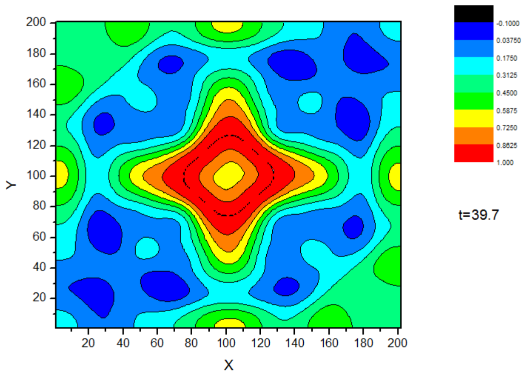

(b)

Fig. 5.4. Elliptical ring solitons at t=39.7.

We can see from figures above, both circular ring soliton and elliptical ring soliton experience shrink (e.g. Figure 4.2 and Figure 5.2) and expansion (e.g. Figure 4.3 and Figure 5.3) and the symmetry of initial conditions are to some extent retained even after evolving for a long time (e.g. Figure 4.5 and Figure 5.4).

4. Conclusion

We have discussed some of the behaviors of solitons represented by soliton solutions of KdV equation and SGE in detail. From the discussions above, we can draw the following conclusions:

i. Solitary waves can be generated from a two-level waveform (e.g. a sudden change in the level of an ocean floor in reality);

ii. Solitons with larger amplitudes travel faster than those with smaller amplitudes, and the velocities of solitons are constant during propagation.

iii. After collision, solitons retain their initial waveforms and velocities.

iv. Superposition of solitons is not linear due to their nonlinear properties.

v. Both circular and elliptical ring solitons (quasi-solitons) would experience shrink and expansion, and their symmetries are to some extent retained during evolution.

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